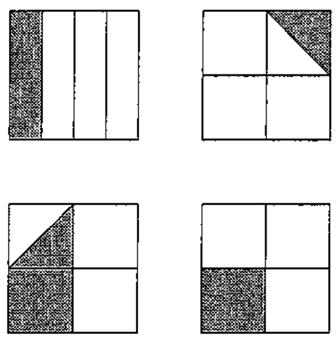
Guide to Student Practice Questions

Each of the following four large congruent squares is subdivided into combinations of congruent triangles or rectangles and is partially shaded. What percent of the total area is partially shaded?



(A) $12\frac{1}{2}$ (B) 20 (C) 25 (D) $33\frac{1}{3}$ (E) $37\frac{1}{2}$

The original problem and choices from the 2011 AMC 8 contest

2011 AMC 8, Problem #7—
"Find the shaded portion of each square separately. "

Problem number
Hint

Solution

Answer (C): The upper left and the lower right squares are each one-fourth shaded, for a total of one-half square. The shaded portions of the upper right and lower left squares make up one-half square. So the total shaded area is one full square, which is25%of the total area.

Solution from official solutions

Difficulty: Medium
SMP-CCSS: 2, 7
CCSS-M: 6G.1, 6RP.3C

Standards for Math Practice
Common Core State Standard

Difficulty, Percent correct

Easy	100-80%
Med Easy	80-60%
Medium	60-40%
Med Hard	40-20%
Hard	20-0%

10a13-01

A taxi ride costs \$1.50 plus \$0.25 per mile traveled. How much does a 5-mile taxi ride cost?

(A) \$2.25 **(B)** \$2.50 **(C)** \$2.75 **(D)** \$3.00 **(E)** \$3.25

2013 AMC 10A, Problem #1—

"Find the additional costs for 5 miles."

Solution

Answer (C): A 5-mile taxi ride costs $\$1.50 + 5(\$0.25) = \$2.75$.

Difficulty: Easy

SMP-CCSS: 4. Model with mathematics.
CCSS-M: F-LE. Construct and compare linear, quadratic, and exponential models and solve problems.

10a13-02

Alice is making a batch of cookies and needs $2\frac{1}{2}$ cups of sugar. Unfortunately, her measuring cup holds only $\frac{1}{4}$ cup of sugar. How many times must she fill that cup to get the correct amount of sugar?

(A) 8 **(B)** 10 **(C)** 12 **(D)** 16 **(E)** 20

2013 AMC 10A, Problem #2—

"How many times must she fill the cup to get 1 cup of sugar?"

Solution

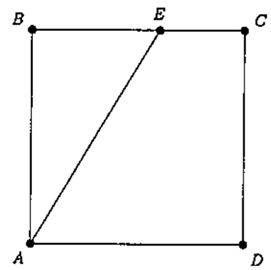
Answer (B): Filling the cup 4 times will give Alice 1 cup of sugar. To get $2\frac{1}{2}$ cups of sugar, she must fill it $4 + 4 + \frac{1}{2} \cdot 4 = 10$ times.

Difficulty: Easy

SMP-CCSS: 4. Model with mathematics.
CCSS-M: N-Q. Reason quantitatively and use units to solve problems.

10a13-03, 12a13-01

Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?



(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

2013 AMC 10A, Problem #3—
2013 AMC 12A, Problem #1—

"Use the formula for area of a triangle."

Solution

Answer (E): The legs of $\triangle ABE$ have lengths $AB = 10$ and BE. Therefore $\frac{1}{2} \cdot 10 \cdot BE = 40$, so $BE = 8$.

Difficulty: Easy

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

10a13-04, 12a13-02

A softball team played ten games, scoring 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 runs. They lost by one run in exactly five games. In each of their other games, they scored twice as many runs as their opponent. How many total runs did their opponents score?

(A) 35 **(B)** 40 **(C)** 45 **(D)** 50 **(E)** 55

2013 AMC 10A, Problem #4—
2013 AMC 12A, Problem #2—

"Find the number of runs scored by the opponents when the softball team scored an even number of runs."

Solution

Answer (C): The softball team could only have scored twice as many runs as their opponent when they scored an even number of runs. In those games their opponents scored

$$\frac{2}{2}+\frac{4}{2}+\frac{6}{2}+\frac{8}{2}+\frac{10}{2}=15 \text{ runs.}$$

In the games the softball team lost, their opponents scored

$$(1+1)+(3+1)+(5+1)+(7+1)+(9+1)=30 \text{ runs.}$$

The total number of runs scored by their opponents was $15+30=45$ runs.

Difficulty: Medium Easy

SMP-CCSS:
CCSS-M:

10a13-05, 12a13-05

Tom, Dorothy, and Sammy went on a vacation and agreed to split the costs evenly. During their trip Tom paid \$105, Dorothy paid \$125, and Sammy paid \$175. In order to share the costs equally, Tom gave Sammy t dollars, and Dorothy gave Sammy d dollars. What is $t - d$?

(A) 15 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 35

2013 AMC 10A, Problem #5—
2013 AMC 12A, Problem #5—

"What is the fair share of each traveler?"

Solution

Answer (B): The total shared expenses were $105 + 125 + 175 = 405$ dollars, so each traveler's fair share was $\frac{1}{3} \cdot 405 = 135$ dollars. Therefore $t = 135 - 105 = 30$ and $d = 135 - 125 = 10$, so $t - d = 30 - 10 = 20$.

OR

Because Dorothy paid 20 dollars more than Tom, Sammy must receive 20 more dollars from Tom than from Dorothy.

Difficulty: Easy

SMP-CCSS: 4. Model with mathematics.
CCSS-M:

10a13-06

Joey and his five brothers are ages 3, 5, 7, 9, 11, and 13. One afternoon two of his brothers whose ages sum to 16 went to the movies, two brothers younger than 10 went to play baseball, and Joey and the 5-year-old stayed home. How old is Joey?

(A) 3 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

2013 AMC 10A, Problem #6—

"Which brothers went to the movies?"

Solution

Answer (D): The 5-year-old and the two brothers who went to play baseball account for three of the four brothers who are younger than 10. Because the only age pairs that sum to 16 are 3 and 13, 5 and 11, and 7 and 9, the brothers who went to the movies must be 3 and 13 years old. Hence the 7-year-old and 9-year-old brothers went to play baseball, and Joey is 11.

Difficulty: Easy

SMP-CCSS: 3. Construct viable arguments and critique the reasoning of others.
CCSS-M:

10a13-07

A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 16

2013 AMC 10A, Problem #7—

"How many possible choices are there for the final course(s)?"

Solution

Answer (C): Because English is required, the student must choose 3 of the remaining 5 courses. If the student takes both math courses, there are 3 possible choices for the final course. If the student chooses only one of the 2 possible math courses, then the student must omit one of the 3 remaining courses, for a total of $2 \cdot 3 = 6$ programs. Hence there are $3 + 6 = 9$ programs.

OR

Because English is required, there are 5 remaining courses from which a student must choose 3. Of those $\binom{5}{3}$ possibilities, one does not include a math course. Thus the number of possible programs is $\binom{5}{3} - 1 = 9$.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: S-CP.9. Use permutations and combinations to compute probabilities of compound events and solve problems.

10a13-08, 12a13-04

What is the value of

$$\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}}?$$

(A) -1 **(B)** 1 **(C)** $\frac{5}{3}$ **(D)** 2013 **(E)** 2^{4024}

2013 AMC 10A, Problem #8—
2013 AMC 12A, Problem #4—

"Factor 2^{2012} from each term."

Solution

Answer (C): Factoring 2^{2012} from each of the terms and simplifying gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)}=\frac{4+1}{4-1}=\frac{5}{3}.$$

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: N-RN. Extend the properties of exponents to rational exponents.

10a13-09, 12a13-06

In a recent basketball game, Shenille attempted only three-point shots and two-point shots. She was successful on 20% of her three-point shots and 30% of her two-point shots. Shenille attempted 30 shots. How many points did she score?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 36

2013 AMC 10A, Problem #9—
2013 AMC 12A, Problem #6—

"Let the number of attempted three-point shots be x."

Solution

Answer (B): If Shenille attempted x three-point shots and $30 - x$ two-point shots, then she scored a total of $\frac{20}{100} \cdot 3 \cdot x + \frac{30}{100} \cdot 2 \cdot (30 - x) = 18$ points.
Remark: The given information does not allow the value of x to be determined.

Difficulty: Medium

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: A-CED. Create equations that describe numbers or relationships.

10a13-10, 12a13-03

A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

(A) 15 **(B)** 30 **(C)** 40 **(D)** 60 **(E)** 70

2013 AMC 10A, Problem #10—
2013 AMC 12A, Problem #3—

"How many pink carnations and red carnations are there?"

Solution

Answer (E): Because six tenths of the flowers are pink and two thirds of the pink flowers are carnations, $\frac{6}{10} \cdot \frac{2}{3} = \frac{2}{5}$ of the flowers are pink carnations. Because four tenths of the flowers are red and three fourths of the red flowers are carnations, $\frac{4}{10} \cdot \frac{3}{4} = \frac{3}{10}$ of the flowers are red carnations. Therefore $\frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$ of the flowers are carnations.

Difficulty: Medium Easy

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: A-REI. Understand solving equations as a process of reasoning and explain the reasoning.

10a13-11

A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 25

2013 AMC 10A, Problem #11—

"What is the number of student council members?"

Solution

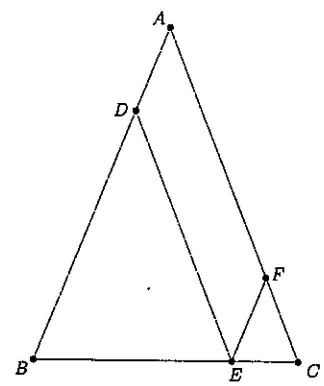
Answer (A): Let n be the number of student council members. Because there are 10 ways of choosing the two-person welcoming committee, it follows that $10 = \binom{n}{2} = \frac{1}{2}n(n-1)$, from which $n = 5$. The number of ways to select the three-person planning committee is $\binom{5}{3} = 10$.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: S-CP.9. Use permutations and combinations to compute probabilities of compound events and solve problems.

10a13-12, 12a13-09

In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points D, E, and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?



(A) 48 **(B)** 52 **(C)** 56 **(D)** 60 **(E)** 72

2013 AMC 10A, Problem #12—
2013 AMC 12A, Problem #9—

"Find similar triangles."

Solution

Answer (C): Because $\overline{EF}$ is parallel to $\overline{AB}$, it follows that $\triangle FEC$ is similar to $\triangle ABC$ and $FE = FC$. Thus half of the perimeter of $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

Difficulty: Medium

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT.2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar.

10a13-13

How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

(A) 52 **(B)** 60 **(C)** 66 **(D)** 68 **(E)** 70

2013 AMC 10A, Problem #13—

"What form must the three-digit number have?"

Solution

Answer (B): Each such three-digit number must have the form aba, where a and b are digits and $a \neq 0$. Such a number will not be divisible by 5 if and only if $a \neq 5$. If a is equal to 1, 2, 3, or 4, then any of the ten choices for b satisfies the requirement. If a is equal to 6, 7, 8, or 9, then there are 8, 6, 4, or 2 choices for b, respectively. This results in $4 \cdot 10 + 8 + 6 + 4 + 2 = 60$ numbers.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

10a13-14

A solid cube of side length 1 is removed from each corner of a solid cube of side length 3. How many edges does the remaining solid have?

(A) 36 **(B)** 60 **(C)** 72 **(D)** 84 **(E)** 108

2013 AMC 10A, Problem #14—

"How many edges does the initial large cube have?"

Solution

Answer (D): The large cube has 12 edges, and a portion of each edge remains after the 8 small cubes are removed. All of the 12 edges of each small cube are also edges of the new solid, except for the 3 edges that meet at a vertex of the large cube. Thus the new solid has a total of $12 + 8(12 - 3) = 84$ edges.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-GMD. Visualize relationships between two-dimensional and three-dimensional objects.

10a13-15

Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 18

2013 AMC 10A, Problem #15—

"What is the average of the lengths of the altitudes to the two given sides?"

Solution

Answer (D): Denote the length of the third side as x, and the altitudes to the sides of lengths 10 and 15 as m and n, respectively. Then twice the area of the triangle is $10m = 15n = \frac{1}{2}x(m+n)$. This implies that $m = \frac{3}{2}n$, so

$$15n = \frac{1}{2}x\left(\frac{3}{2}n + n\right) = \frac{5}{4}xn.$$

Therefore $15 = \frac{5}{4}x$, and $x = 12$.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT. Understand similarity in terms of similarity transformations.

10a13-16

A triangle with vertices $(6,5)$, $(8,-3)$, and $(9,1)$ is reflected about the line $x = 8$ to create a second triangle. What is the area of the union of the two triangles?

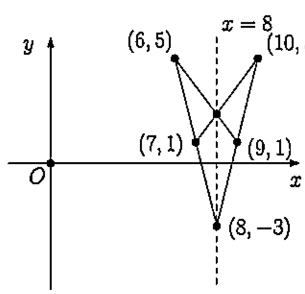
(A) 9 **(B)** $\frac{28}{3}$ **(C)** 10 **(D)** $\frac{31}{3}$ **(E)** $\frac{32}{3}$

2013 AMC 10A, Problem #16—

"What are the vertices of the reflected triangle?"

Solution

Answer (E): The reflected triangle has vertices $(7,1)$, $(8,-3)$, and $(10,5)$. The point $(9,1)$ is on the line segment from $(10,5)$ to $(8,-3)$. The line segment from $(6,5)$ to $(9,1)$ contains the point $(8,\frac{7}{3})$, which must be on both triangles, and by symmetry the point $(7,1)$ is on the line segment from $(6,5)$ to $(8,-3)$. Therefore the union of the two triangles is also the union of two congruent triangles with disjoint interiors, each having the line segment from $(8,-3)$ to $(8,\frac{7}{3})$ as a base. The altitude of one of the two triangles is the distance from the line $x = 8$ to the point $(10,5)$, which is 2. Hence the union of the triangles has area $2 \cdot (\frac{1}{2} \cdot 2 \cdot (\frac{7}{3} + 3)) = \frac{32}{3}$.



Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-CO. Experiment with transformations in the plane.

10a13-17

Daphne is visited periodically by her three best friends: Alice, Beatrix, and Claire. Alice visits every third day, Beatrix visits every fourth day, and Claire visits every fifth day. All three friends visited Daphne yesterday. How many days of the next 365-day period will exactly two friends visit her?

(A) 48 **(B)** 54 **(C)** 60 **(D)** 66 **(E)** 72

2013 AMC 10A, Problem #17—

"Find the number of times that each pair of friends will visit Daphne in a 365-day period."

Solution

Answer (B): Alice and Beatrix will visit Daphne together every $3 \cdot 4 = 12$ days, so this will happen $\lfloor \frac{365}{12} \rfloor = 30$ times. Likewise Alice and Claire will visit together $\lfloor \frac{365}{3 \cdot 5} \rfloor = 24$ times, and Beatrix and Claire will visit together $\lfloor \frac{365}{4 \cdot 5} \rfloor = 18$ times. However, each of these counts includes the $\lfloor \frac{365}{3 \cdot 4 \cdot 5} \rfloor = 6$ times when all three friends visit. The number of days that exactly two friends visit is $(30 - 6) + (24 - 6) + (18 - 6) = 54$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: A-SSE. Write expressions in equivalent forms to solve problems.

10a13-18, 12a13-13

Let points $A = (0, 0)$, $B = (1, 2)$, $C = (3, 3)$, and $D = (4, 0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $(\frac{p}{q}, \frac{r}{s})$, where these fractions are in lowest terms. What is $p+q+r+s$?

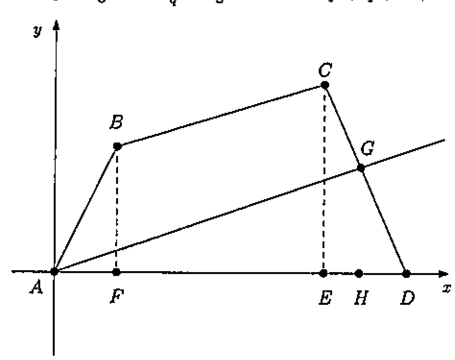
(A) 54 **(B)** 58 **(C)** 62 **(D)** 70 **(E)** 75

2013 AMC 10A, Problem #18—
2013 AMC 12A, Problem #13—

"Find the area of $ABCD$."

Solution

Answer (B): Let line AG be the required line, with G on $\overline{CD}$. Divide $ABCD$ into triangle ABF, trapezoid $BCEF$, and triangle CDE, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x-4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p + q + r + s = 58$.



Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

10a13-19

In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b representation of 2013 end in the digit 3?

(A) 6 **(B)** 9 **(C)** 13 **(D)** 16 **(E)** 18

2013 AMC 10A, Problem #19—

"Find the properties of b."

Solution

Answer (C): For the base-b representation of 2013 to end in the digit 3, the base b must exceed 3. Also, b must divide $2013 - 3 = 2010$, so b must be one of the 16 positive integer factors of $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Thus there are $16 - 3 = 13$ bases in which 2013 ends with a 3.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

10a13-20

A unit square is rotated 45° about its center. What is the area of the region swept out by the interior of the square?

(A) $1-\frac{\sqrt{2}}{2}+\frac{\pi}{4}$ **(B)** $\frac{1}{2}+\frac{\pi}{4}$ **(C)** $2-\sqrt{2}+\frac{\pi}{4}$
(D) $\frac{\sqrt{2}}{2}+\frac{\pi}{4}$ **(E)** $1+\frac{\sqrt{2}}{4}+\frac{\pi}{8}$

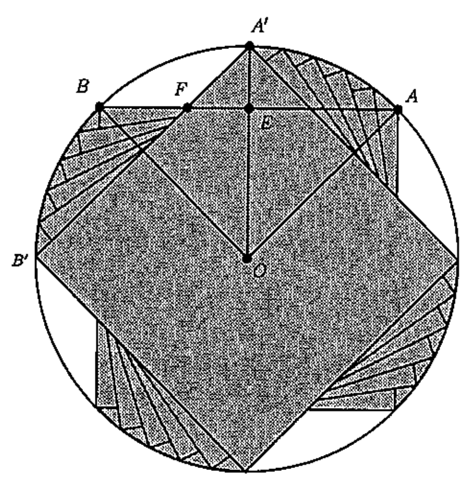
2013 AMC 10A, Problem #20—

"Draw a diagram and find one quarter of the area of the region swept out."

Solution

Answer (C): Let O be the center of unit square $ABCD$, let A and B be rotated to points A' and B', and let $\overline{OA'}$ and $\overline{A'B'}$ intersect $\overline{AB}$ at E and F, respectively. Then one quarter of the region swept out by the interior of the square consists of the 45° sector AOA' with radius $\frac{\sqrt{2}}{2}$, isosceles right triangle OEB with leg length $\frac{1}{2}$, and isosceles right triangle $A'EF$ with leg length $\frac{\sqrt{2}-1}{2}$. Thus the area of the region is

$$4\left(\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}-1}{2}\right)^2\right)=2-\sqrt{2}+\frac{\pi}{4}.$$



Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-C.4. Construct a tangent line from a point outside a given circle to the circle.

10a13-21, 12a13-17

A group of 12 pirates agree to divide a treasure chest of gold coins among themselves as follows. The k^{th} pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the 12^{th} pirate receive?

(A) 720 **(B)** 1296 **(C)** 1728 **(D)** 1925 **(E)** 3850

2013 AMC 10A, Problem #21—
2013 AMC 12A, Problem #17—

"What is the number of coins originally in the chest?"

Solution

Answer (D): For $1 \le k \le 11$, the number of coins remaining in the chest before the k^{th} pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are n coins left for the 12^{th} pirate to take, the number of coins originally in the chest is

$$\frac{12^{11} \cdot n}{11!} = \frac{2^{22} \cdot 3^{11} \cdot n}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} = \frac{2^{14} \cdot 3^7 \cdot n}{5^2 \cdot 7 \cdot 11}.$$

The smallest value of n for which this is a positive integer is $5^2 \cdot 7 \cdot 11 = 1925$.
In this case there are

$$2^{14} \cdot 3^7 \cdot \frac{11!}{(12-k)! \cdot 12^{k-1}}$$

coins left for the k^{th} pirate to take, and note that this amount is an integer for each k. Hence the 12^{th} pirate receives 1925 coins.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

10a13-22, 12a13-18

Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

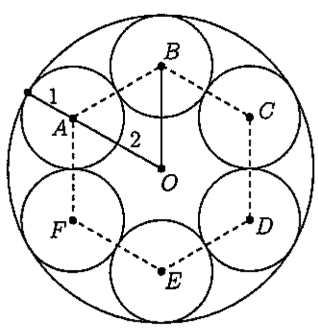
(A) $\sqrt{2}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** $\sqrt{3}$ **(E)** 2

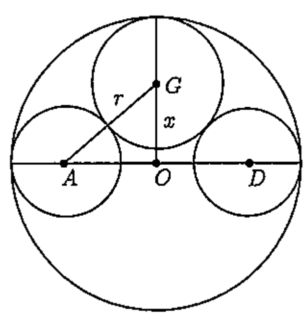
2013 AMC 10A, Problem #22—
2013 AMC 12A, Problem #18—

"What is the radius of the largest sphere?"

Solution

Answer (B): Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at O.





Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-GMD.4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects.

10a13-23, 12a13-19

In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

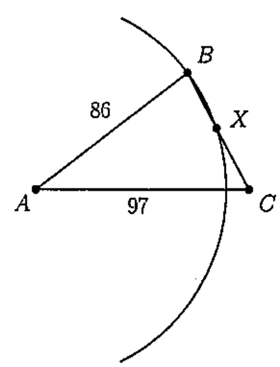
(A) 11 **(B)** 28 **(C)** 33 **(D)** 61 **(E)** 72

2013 AMC 10A, Problem #23—
2013 AMC 12A, Problem #19—

"Find $BC \cdot CX$."

Solution

Answer (D): By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and CX are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.



Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-C.2. Identify and describe relationships among inscribed angles, radii, and chords.

10a13-24

Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?

(A) 540 **(B)** 600 **(C)** 720 **(D)** 810 **(E)** 900

2013 AMC 10A, Problem #24—

"What is the number of possible strings for the players in each school?"

Solution

Answer (E): Call the players from Central A, B, and C, and call the players from Northern X, Y, and Z. Represent the schedule for each Central player by a string of length six consisting of two each of X, Y, and Z. There are $\binom{6}{2}\binom{4}{2} = 90$ possible strings for player A. Assume without loss of generality that the string is $XXYYZZ$. Player B's schedule must be a string with no X's in the first two positions, no Y's in the next two, and no Z's in the last two. If B's string begins with a Y and a Z in either order, the next two letters must be an X and a Z, and the last two must be an X and a Y. Because each pair can be ordered in one of two ways, there are $2^3 = 8$ such strings. If B's string begins with YY or ZZ, it must be $YYZZXX$ or $ZZXXYY$, respectively. Hence there are 10 possible schedules for B for each of the 90 schedules for A, and C's schedule is then determined. The total number of possible schedules is 900.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

10a13-25

All 20 diagonals are drawn in a regular octagon. At how many distinct points in the interior of the octagon (not on the boundary) do two or more diagonals intersect?

(A) 49 **(B)** 65 **(C)** 70 **(D)** 96 **(E)** 128

2013 AMC 10A, Problem #25—

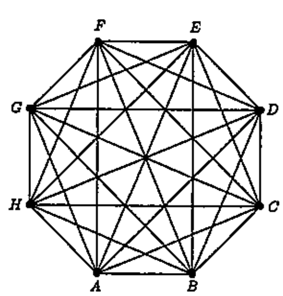
"Draw and label an octagon and determine the types of diagonals which exist."

Solution

Answer (A): Label the octagon $ABCDEFGH$. There are 20 diagonals in all, 5 with endpoints at each vertex. The diagonals are of three types:

- Diagonals that skip over only one vertex, such as $\overline{AC}$ or $\overline{AG}$. These diagonals intersect with each of the five diagonals with endpoints at the skipped vertex.
- Diagonals that skip two vertices, such as $\overline{AD}$ or $\overline{AF}$. These diagonals intersect with four of the five diagonals that have endpoints at each of the two skipped vertices.
- Diagonals that cross to the opposite vertex, such as $\overline{AE}$. These diagonals intersect with three of the five diagonals that have endpoints at each of the three skipped vertices.

Therefore, from any given vertex, the diagonals will intersect other diagonals at $2 \cdot 5 + 2 \cdot 8 + 1 \cdot 9 = 35$ points. Counting from all 8 vertices, the total is $8 \cdot 35 = 280$ points.
Observe that, by symmetry, all four diagonals that cross to the opposite vertex intersect in the center of the octagon. This single intersection point has been counted 24 times, 3 from each of the 8 vertices. Further observe that at each of the vertices of the smallest internal octagon created by the diagonals, 3 diagonals intersect. For example, $\overline{AD}$ intersects with $\overline{CH}$ on $\overline{BF}$. These 8 intersection points have each been counted 12 times, 2 from each of the 6 affected vertices.The remaining intersection points each involve only two diagonals and each has been counted 4 times, once from each endpoint. These number $\frac{280-24-8\cdot 12}{4} = 40$. There are therefore $1 + 8 + 40 = 49$ distinct intersection points in the interior of the octagon.



Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

10b13-01

What is $\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6}$?

(A) -1 **(B)** $\frac{5}{36}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{49}{20}$ **(E)** $\frac{43}{3}$

2013 AMC 10B, Problem #1—

"Simplify the terms."

Solution

Answer (C): Simplifying gives

$$\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6} = \frac{12}{9} - \frac{9}{12} = \frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} = \frac{7}{12}.$$

Difficulty: Easy

SMP-CCSS: 6. Attend to precision; 8. Look for and express regularity in repeated reasoning.
CCSS-M: N-RN. Use properties of rational and irrational numbers.

10b13-02, 12b13-02

Mr. Green measures his rectangular garden by walking two of the sides and finds that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden?

(A) 600 **(B)** 800 **(C)** 1000 **(D)** 1200 **(E)** 1400

2013 AMC 10B, Problem #2—
2013 AMC 12B, Problem #2—

"What are the width and length of the garden?"

Solution

Answer (A): The garden is $2 \cdot 15 = 30$ feet wide and $2 \cdot 20 = 40$ feet long. Hence Mr. Green expects $\frac{1}{2} \cdot 30 \cdot 40 = 600$ pounds of potatoes.

Difficulty: Easy

SMP-CCSS: 2. Reason abstractly and quantitatively; 4. Model with mathematics.
CCSS-M: N-Q. Reason quantitatively and use units to solve problems.

10b13-03, 12b13-01

On a particular January day, the high temperature in Lincoln, Nebraska, was 16 degrees higher than the low temperature, and the average of the high and low temperatures was 3°. In degrees, what was the low temperature in Lincoln that day?

(A) -13 **(B)** -8 **(C)** -5 **(D)** -3 **(E)** 11

2013 AMC 10B, Problem #3—
2013 AMC 12B, Problem #1—

"Find the difference between the high and low temperatures and the average temperature."

Solution

Answer (C): The difference between the high and low temperatures was 16 degrees, so the difference between each of these and the average temperature was 8 degrees. The low temperature was 8 degrees less than the average, so it was $3^\circ - 8^\circ = -5^\circ$.

Difficulty: Easy

SMP-CCSS: 4. Model with mathematics.
CCSS-M: N-RN. Use properties of rational and irrational numbers.

10b13-04, 12b13-03

When counting from 3 to 201, 53 is the 51^{st} number counted. When counting backwards from 201 to 3, 53 is the n^{th} number counted. What is n?

(A) 146 **(B)** 147 **(C)** 148 **(D)** 149 **(E)** 150

2013 AMC 10B, Problem #4—
2013 AMC 12B, Problem #3—

"Find a pattern for the numbers counted backwards."

Solution

Answer (D): The number 201 is the 1^{st} number counted when proceeding backwards from 201 to 3. In turn, 200 is the 2^{nd} number, 199 is the 3^{rd} number, and x is the $(202-x)^{th}$ number. Therefore 53 is the $(202-53)^{th}$ number, which is the 149^{th} number.

Difficulty: Medium

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 2. Reason abstractly and quantitatively; 3. Construct viable arguments and critique the reasoning of others.
CCSS-M:

10b13-05

Positive integers a and b are each less than 6. What is the smallest possible value for $2 \cdot a - a \cdot b$?

(A) -20 **(B)** -15 **(C)** -10 **(D)** 0 **(E)** 2

2013 AMC 10B, Problem #5—

"Factorize $2 \cdot a - a \cdot b$**."**

Solution

Answer (B): Note that $2 \cdot a - a \cdot b = (2 - b)a$. This expression is negative when $b > 2$. Hence the product is minimized when a and b are as large as possible. The minimum value is $(2 - 5) \cdot 5 = -15$.

Difficulty: Medium Easy

SMP-CCSS: 2. Reason abstractly and quantitatively; 3. Construct viable arguments and critique the reasoning of others; 4. Model with mathematics.
CCSS-M: F-IF. Analyze functions using different representations; A-REI. Understand solving equations as a process of reasoning and explain the reasoning.

10b13-06, 12b13-05

The average age of 33 fifth-graders is 11. The average age of 55 of their parents is 33. What is the average age of all of these parents and fifth-graders?

(A) 22 **(B)** 23.25 **(C)** 24.75 **(D)** 26.25 **(E)** 28

2013 AMC 10B, Problem #6—
2013 AMC 12B, Problem #5—

"What is the sum of all the ages?"

Solution

Answer (C): The sum of all the ages is $55 \cdot 33 + 33 \cdot 11 = 33 \cdot 66$, so the average of all the ages is

$$\frac{33 \cdot 66}{55 + 33} = \frac{33 \cdot 66}{88} = \frac{33 \cdot 3}{4} = 24.75\,.$$

Difficulty: Medium Easy

SMP-CCSS: 2. Reason abstractly and quantitatively; 4. Model with mathematics.
CCSS-M: S-ID. Summarize, represent, and interpret data on a single count or measurement variable.

10b13-07

Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle?

(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** 1 **(D)** $\sqrt{2}$ **(E)** 2

2013 AMC 10B, Problem #7—

"Use the Inscribed Angle Theorem."

Solution

Answer (B): The six points divide the circle into six arcs each measuring $60°$. By the Inscribed Angle Theorem, the angles of the triangle can only be $30°$, $60°$, $90°$, and $120°$. Because the angles of the triangle are pairwise distinct the triangle must be a $30-60-90°$ triangle. Therefore the hypotenuse of the triangle is the diameter of the circle, and the legs have lengths 1 and $\sqrt{3}$. The area of the triangle is $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$.

Difficulty: Medium

SMP-CCSS: 5. Use appropriate tools strategically; 7. Look for and make use of structure.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

10b13-08, 12b13-04

Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?

(A) 10 **(B)** 16 **(C)** 25 **(D)** 30 **(E)** 40

2013 AMC 10B, Problem #8—
2013 AMC 12B, Problem #4—

"Find the number of gallons of gasoline used by Ray's car and Tom's car respectively."

Solution

Answer (B): Let D equal the distance traveled by each car. Then Ray's car uses $\frac{D}{40}$ gallons of gasoline and Tom's car uses $\frac{D}{10}$ gallons of gasoline. The cars combined miles per gallon of gasoline is

$$\frac{2D}{\left(\frac{D}{40}+\frac{D}{10}\right)} = 16.$$

Difficulty: Medium Hard

SMP-CCSS: 4. Model with mathematics.
CCSS-M: N-Q. Reason quantitatively and use units to solve problems.

10b13-09

Three positive integers are each greater than 1, have a product of 27,000, and are pairwise relatively prime. What is the sum of these integers?

(A) 100 **(B)** 137 **(C)** 156 **(D)** 160 **(E)** 165

2013 AMC 10B, Problem #9—

"Find the pairwise relatively prime positive integers."

Solution

Answer (D): Note that

$$27{,}000 = 2^3 \cdot 3^3 \cdot 5^3.$$

The only three pairwise relatively prime positive integers greater than 1 with a product of 27,000 are 8, 27, and 125. The sum of these numbers is 160.

Difficulty: Medium Hard

SMP-CCSS: 8. Look for and express regularity in repeated reasoning.
CCSS-M:

10b13-10

A basketball team's players were successful on 50% of their two-point shots and 40% of their three-point shots, which resulted in 54 points. They attempted 50% more two-point shots than three-point shots. How many three-point shots did they attempt?

(A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

2013 AMC 10B, Problem #10—

"Find the number of three-point shots attempted."

Solution

Answer (C): Let x denote the number of three-point shots attempted. Then the number of three-point shots made was $0.4x$, resulting in $3(0.4x) = 1.2x$ points. The number of two-point shots attempted was $1.5x$, and they were successful on $0.5(1.5x) = 0.75x$ of them resulting in $2(0.75x) = 1.5x$ points. The number of points scored was $1.2x + 1.5x = 54$, so $x = 20$.

Difficulty: Medium

SMP-CCSS: 2. Reason abstractly and quantitatively.
CCSS-M: A-CED. Create equations that describe numbers or relationships.

10b13-11, 12b13-06

Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 8

2013 AMC 10B, Problem #11—
2013 AMC 12B, Problem #6—

"Complete the squares in the equation."

Solution

Answer (B): By completing the square the equation can be rewritten as follows:

$$x^2 + y^2 = 10x - 6y - 34,$$

$$x^2 - 10x + 25 + y^2 + 6y + 9 = 0,$$

$$(x-5)^2 + (y+3)^2 = 0.$$

Therefore $x = 5$ and $y = -3$, so $x + y = 2$.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure.
CCSS-M: A-SSE. Write expressions in equivalent forms to solve problems; A-REI. Solve systems of equations.

10b13-12

Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?

(A) $\frac{2}{5}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{4}{5}$

2013 AMC 10B, Problem #12—

"The five sides of the pentagon are congruent."

Solution

Answer (B): The five sides of the pentagon are congruent, and the five congruent diagonals are longer than the sides. Once one segment is selected, 4 of the 9 remaining segments have the same length as that segment. Therefore the requested probability is $\frac{4}{9}$.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: S-CP.1. Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (or, and, not).

10b13-13, 12b13-07

Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1", so Blair follows by saying "1, 2". Jo then says "1, 2, 3", and so on. What is the 53^{rd} number said?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 8

2013 AMC 10B, Problem #13—
2013 AMC 12B, Problem #7—

"Find a pattern for the numbers that have been said."

Solution

Answer (E): Note that Jo starts by saying 1 number, and this is followed by Blair saying 2 numbers, then Jo saying 3 numbers, and so on. After someone completes her turn after saying the number n, then $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ numbers have been said. If $n = 9$ then 45 numbers have been said. Therefore there are $53 - 45 = 8$ more numbers that need to be said. The 53^{rd} number said is 8.

Difficulty: Easy

SMP-CCSS: 8. Look for and express regularity in repeated reasoning.
CCSS-M:

10b13-14

Define $a\clubsuit b = a^2b - ab^2$. Which of the following describes the set of points (x, y) for which $x\clubsuit y = y\clubsuit x$?

(A) a finite set of points
(B) one line
(C) two parallel lines
(D) two intersecting lines
(E) three lines

2013 AMC 10B, Problem #14—

"Factorize the equation $x\clubsuit y = y\clubsuit x$."

Solution

Answer (E): The equation $x\clubsuit y = y\clubsuit x$ is equivalent to $x^2y - xy^2 = y^2x - yx^2$. This equation is euivalent to gives $2xy(x - y) = 0$. This equation will hold exactly if $x = 0$, $y = 0$, or $x = y$. The solution set consists of three lines: the x-axis, the y-axis, and the line $x = y$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure.
CCSS-M: A-REI. Solve systems of equations.

10b13-15

A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

(A) 1 **(B)** $\frac{\sqrt{6}}{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $\frac{3\sqrt{2}}{2}$

2013 AMC 10B, Problem #15—

"Subdivide the hexagon into 6 equilateral triangles."

Solution

Answer (B): Let s be the side length of the triangle and h the side length of the hexagon. The hexagon can be subdivided into 6 equilateral triangles by drawing segments from the center of the hexagon to each vertex. Because the areas of the large triangle and hexagon are equal, the triangles in the hexagon each have area $\frac{1}{6}$ of the area of the large triangle. Thus

$$\frac{h}{s} = \sqrt{\frac{1}{6}} \quad \text{so} \quad h = \frac{\sqrt{6}}{6}s.$$

The perimeter of the triangle is $a = 3s$ and the perimeter of the hexagon is $b = 6h = \sqrt{6}s$, so

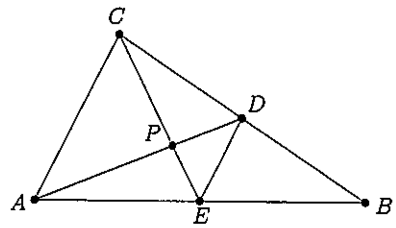
$$\frac{a}{b} = \frac{3s}{\sqrt{6}\,s} = \frac{\sqrt{6}}{2}.$$

Difficulty: Medium Hard

SMP-CCSS: 5. Use appropriate tools strategically.
CCSS-M: G-GMD. Visualize relationships between two-dimensional and three-dimensional objects.

10b13-16

In $\triangle ABC$, medians $\overline{AD}$ and $\overline{CE}$ intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



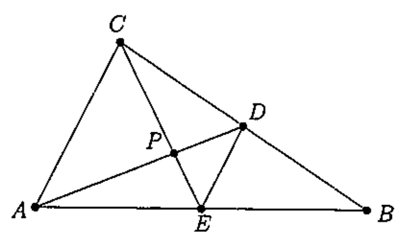
(A) 13 **(B)** 13.5 **(C)** 14 **(D)** 14.5 **(E)** 15

2013 AMC 10B, Problem #16—

"$\triangle DPE$ is a right triangle."

Solution

Answer (B): The ratio of $PE{:}PD{:}DE$ is 3:4:5.



Hence by the converse of the Pythagorean Theorem, $\triangle DPE$ is a right triangle. Therefore $\overline{CE}$ is perpendicular to $\overline{AD}$, and the area of $AEDC$ is one-half the product of its diagonals. Because P is the centroid of $\triangle ABC$, it follows that $CE = 3(PE) = 4.5$ and $AD = 3(PD) = 6$. Therefore the area of $AEDC$ is $0.5(4.5)(6) = 13.5$.

OR

From the first solution, triangles CPD, DPE, EPA, and APC are right triangles with right angle at P. The area of trapezoid $AEDC$ is given by the sum of the areas of these four triangles. Because $\overline{DE}$ is parallel to $\overline{AC}$ and D is the midpoint of $\overline{CB}$, triangles BAC and BED are similar with common ratio 2, so $AC = 2 \cdot DE = 5$. Triangles APC and DPE are similar, so $AP = 4$ and $CP = 3$. Thus the area of $AEDC$ is

$$\frac{1}{2} \cdot 4 \cdot 1.5 + \frac{1}{2} \cdot 3 \cdot 4 + \frac{1}{2} \cdot 2 \cdot 3 + \frac{1}{2} \cdot 2 \cdot 1.5 = 13.5\,.$$

Difficulty: Medium Hard

SMP-CCSS: 5. Use appropriate tools strategically.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

10b13-17, 12b13-10

Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

(A) 62 **(B)** 82 **(C)** 83 **(D)** 102 **(E)** 103

2013 AMC 10B, Problem #17—
2013 AMC 12B, Problem #10—

"No more exchanges are possible when Alex has fewer than 2 red tokens and fewer than 3 blue tokens."

Solution

Answer (E): After Alex makes m exchanges at the first booth and n exchanges at the second booth, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. No more exchanges are possible when he has fewer than 2 red tokens and fewer than 3 blue tokens. Therefore no more exchanges are possible if and only if $2m-n \geq 74$ and $3n-m \geq 73$. Equality can be achieved when $(m,n)=(59,44)$, and Alex will have $59+44=103$ silver tokens.
Note that the following exchanges produce 103 silver tokens:

	Red Tokens	Blue Tokens	Silver Tokens
Exchange 75 blue tokens	100	0	25
Exchange 100 red tokens	0	50	75
Exchange 48 blue tokens	16	2	91
Exchange 16 red tokens	0	10	99
Exchange 9 blue tokens	3	1	102
Exchange 2 red tokens	1	2	103

Difficulty: Medium

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure.
CCSS-M:

10b13-18

The number 2013 has the property that its units digit is the sum of its other digits, that is $2 + 0 + 1 = 3$. How many integers less than 2013 but greater than 1000 share this property?

(A) 33 **(B)** 34 **(C)** 45 **(D)** 46 **(E)** 58

2013 AMC 10B, Problem #18—

"Consider the numbers in the ranges 1001 to 1999 and 2000 to 2013."

Solution

Answer (D): First note that the only number between 2000 and 2013 that shares this property is 2002. Consider now the numbers in the range 1001 to 1999. There is exactly 1 number, 1001, that shares the property when the units digits is 1. There are exactly 2 numbers, 1102 and 1012, when the units digit is 2; exactly 3 numbers, 1203, 1113, and 1023, when the units digits is 3, and so on. Because the thousands digit is always 1, when the units digit is n, for $1 \le n \le 9$, the sum of the hundreds and tens digits must be $n - 1$. There are exactly n ways for this to occur. Hence there are exactly

$$1 + (1 + 2 + \cdots + 9) = 1 + \frac{9 \cdot 10}{2} = 1 + 45 = 46$$

numbers that share this property.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure.
CCSS-M:

10b13-19

The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

(A) $-7 - 4\sqrt{3}$ **(B)** $-2 - \sqrt{3}$ **(C)** -1 **(D)** $-2 + \sqrt{3}$
(E) $-7 + 4\sqrt{3}$

2013 AMC 10B, Problem #19—

"Since the quadratic has exactly one root, $b^2 - 4ac = 0$."

Solution

Answer (D): Let the common difference in the arithmetic sequence be d, so that $a = b + d$ and $c = b - d$. Because the quadratic has exactly one root, $b^2 - 4ac = 0$. Substitution gives $b^2 = 4(b + d)(b - d)$, and therefore $3b^2 = 4d^2$. Because $b \geq 0$ and $d \geq 0$, it follows that $\sqrt{3}b = 2d$. Thus the real root is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} = \frac{-b}{2(b + d)} = \frac{-b}{2\left(b + \frac{\sqrt{3}}{2}b\right)} = -2 + \sqrt{3}.$$

Note that the quadratic equation $x^2 + (4 - 2\sqrt{3})x + 7 - 4\sqrt{3}$ satisfies the given conditions.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: A-REI. Solve systems of equations.

10b13-20, 12b13-15

The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!\cdots a_m!}{b_1!b_2!\cdots b_n!},$$

where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2013 AMC 10B, Problem #20—
2013 AMC 12B, Problem #15—

"What is the prime factorization of 2013?"

Solution

Answer (B): The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. There must be a factor of 61 in the numerator, so $a_1 \geq 61$. Since $a_1!$ will have a factor of 59 and 2013 does not, there must be a factor of 59 in the denominator, and $b_1 \geq 59$. Thus $a_1 + b_1 \geq 120$, and this minimum value can be achieved only if $a_1 = 61$ and $b_1 = 59$. Furthermore, this minimum value is attainable because

$$2013 = \frac{(61!)(11!)(3!)}{(59!)(10!)(5!)}.$$

Thus $|a_1 - b_1| = a_1 - b_1 = 61 - 59 = 2$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure; 8. Look for and express regularity in repeated reasoning.
CCSS-M:

10b13-21, 12b13-14

Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N?

(A) 55 **(B)** 89 **(C)** 104 **(D)** 144 **(E)** 273

2013 AMC 10B, Problem #21—
2013 AMC 12B, Problem #14—

"Find the seventh term as an expression of the first two terms, then use this to find divisibility relations."

Solution

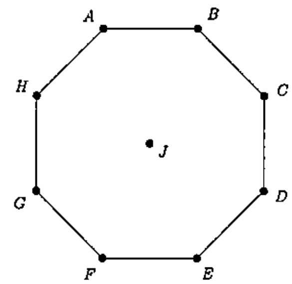
Answer (C): Let the two sequences be (a_n) and (b_n), and assume without loss of generality that $a_1 < b_1$. The definitions of the sequences imply that $a_7 = 5a_1 + 8a_2 = 5b_1 + 8b_2$, so $5(b_1 - a_1) = 8(a_2 - b_2)$. Because 5 and 8 are relatively prime, 8 divides $b_1 - a_1$ and 5 divides $a_2 - b_2$. It follows that $a_1 \le b_1 - 8 \le b_2 - 8 \le a_2 - 13$. The minimum value of N results from choosing $a_1 = 0$, $b_1 = b_2 = 8$, and $a_2 = 13$, in which case $N = 104$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure; 8. Look for and express regularity in repeated reasoning.
CCSS-M: F-IF. Analyze functions using different representations; A-REI. Solve equations and inequalities in one variable.

10b13-22

The regular octagon $ABCDEFGH$ has its center at J. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines AJE, BJF, CJG, and DJH are equal. In how many ways can this be done?



(A) 384 **(B)** 576 **(C)** 1152 **(D)** 1680 **(E)** 3456

2013 AMC 10B, Problem #22—

"What is the sum of all four sums?"

Solution

Answer (C): The digit j at J contributes to all four sums, and each of the other digits contributes to exactly one sum. Therefore the sum of all four sums is $3j + (1 + 2 + 3 + \cdots + 9) = 45 + 3j$. Because all four sums are equal, this must be a multiple of 4, so $j = 1$, 5, or 9. For each choice of j, pair up the remaining digits so that each pair has the same sum. For example, for $j = 1$ the pairs are 2 and 9, 3 and 8, 4 and 7, and 5 and 6. Then order the pairs so that they correspond to the vertex pairs (A, E), (B, F), (C, G), (D, H). This results in $2^4 \cdot 4!$ different combinations for each j. Thus the requirements can be met in $2^4 \cdot 4! \cdot 3 = 1152$ ways.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure; 8. Look for and express regularity in repeated reasoning.
CCSS-M:

10b13-23, 12b13-19

In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

2013 AMC 10B, Problem #23—
2013 AMC 12B, Problem #19—

"Use the Pythagorean Theorem."

Solution

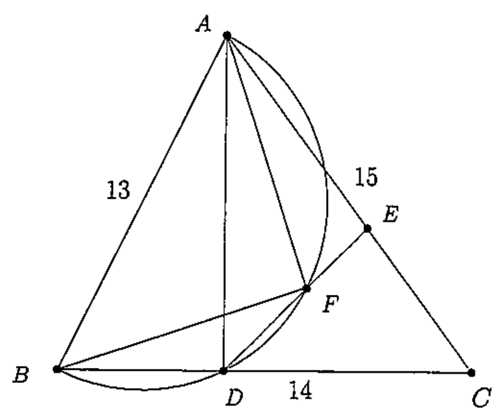
Answer (B): The Pythagorean Theorem applied to right triangles ABD and ACD gives $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$; that is, $13^2 - BD^2 = 15^2 - (14 - BD)^2$, from which it follows that $BD = 5$, $CD = 9$, and $AD = 12$. Because triangles AED and ADC are similar,

$$\frac{AE}{12} = \frac{DE}{9} = \frac{12}{15},$$

implying that $ED = \frac{36}{5}$ and $AE = \frac{48}{5}$.
Because $\angle AFB = \angle ADB = 90°$, it follows that $ABDF$ is cyclic. Thus $\angle ABD + \angle AFD = 180°$ from which $\angle ABD = \angle AFE$. Therefore right triangles ABD and AFE are similar. Hence

$$\frac{FE}{5} = \frac{\frac{48}{5}}{12},$$

from which it follows that $FE = 4$. Consequently $DF = DE - FE = \frac{36}{5} - 4 = \frac{16}{5}$.



Difficulty: Hard
SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: G-SRT. Define trigonometric ratios and solve problems involving right triangles.

10b13-24

A positive integer n is *nice* if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \ldots, 2019\}$ are nice?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2013 AMC 10B, Problem #24—

"Find the condition when integer m has exactly four divisors."

Solution

Answer (A): Let n denote a nice number from the given set. An integer m has exactly four divisors if and only if $m = p^3$ or $m = pq$, where p and q (with $p > q$) are prime numbers. In the former case, the sum of the four divisor is equal to $1 + p + p^2 + p^3$. Note that $1 + 11 + 11^2 + 11^3 < 2010 \leq n$ and $1 + 13 + 13^2 + 13^3 > 2019 \geq n$. Therefore we must have $m = pq$ and $n = 1 + q + p + pq = (1 + q)(1 + p)$. Because p is odd, n must be an even number. If $q = 2$, then n must be divisible by 3. In the given set only $2010 = (1 + 2)(1 + 669)$ and $2016 = (1 + 2)(1 + 671)$ satisfy these requirements. However neither 669 nor 671 are prime. If q is odd, then n must be divisible by 4. In the given set, only 2012 and 2016 are divisible by 4. None of the pairs of factors of 2012, namely $1 \cdot 2012$, $2 \cdot 1006$, $4 \cdot 503$, gives rise to primes p and q. This leaves $2016 = (1 + 3)(1 + 503)$, which is the only nice number in the given set.
Remark: Note that 2016 is nice in five ways. The other four ways are $(1 + 7)(1 + 251)$, $(1 + 11)(1 + 167)$, $(1 + 23)(1 + 83)$, and $(1 + 41)(1 + 47)$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M:

10b13-25, 12b13-23

Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers 10,444 and 3,245, and LeRoy obtains the sum $S = 13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

2013 AMC 10B, Problem #25—
2013 AMC 12B, Problem #23—

"Consider the the last two digits in order of the base-5 representation of N and the last two digits in order of the base-6 representation of N. What relationships can you find among these numbers?"

Solution

Answer (E): Expand the set of three-digit positive integers to include integers N, $0 \le N \le 99$, with leading zeros appended. Because $\text{lcm}(5^2, 6^2, 10^2) = 900$, such an integer N meets the required condition if and only if $N + 900$ does. Therefore N can be considered to be chosen from the set of integers between 000 and 899, inclusive. Suppose that the last two digits in order of the base-5 representation of N are a_1 and a_0. Similarly, suppose that the last two digits of the base-6 representation of N are b_1 and b_0. By assumption, $2N \equiv a_0 + b_0 \pmod{10}$, but $N \equiv a_0 \pmod{5}$ and so

$$a_0 + b_0 \equiv 2N \equiv 2a_0 \pmod{10}.$$

Thus $a_0 \equiv b_0 \pmod{10}$ and because $0 \le a_0 \le 4$ and $0 \le b_0 \le 5$, it follows that $a_0 = b_0$. Because $N \equiv a_0 \pmod{5}$, it follows that there is an integer N_1 such that $N = 5N_1 + a_0$. Also, $N \equiv a_0 \pmod{6}$ implies that $5N_1 + a_0 \equiv a_0 \pmod{6}$ and so $N_1 \equiv 0 \pmod{6}$. It follows that $N_1 = 6N_2$ for some integer N_2 and so $N = 30N_2 + a_0$. Similarly, $N \equiv 5a_1 + a_0 \pmod{25}$ implies that $30N_2 + a_0 \equiv 5a_1 + a_0 \pmod{25}$ and then $N_2 \equiv 6N_2 \equiv a_1 \pmod{5}$. It follows that $N_2 = 5N_3 + a_1$ for some integer N_3 and so $N = 150N_3 + 30a_1 + a_0$. Once more, $N \equiv 6b_1 + a_0 \pmod{36}$ implies that $6N_3 - 6a_1 + a_0 \equiv 150N_3 + 30a_1 + a_0 \equiv 6b_1 + a_0 \pmod{36}$ and then $N_3 \equiv a_1 + b_1 \pmod{6}$. It follows that $N_3 = 6N_4 + a_1 + b_1$ for some integer N_4 and so $N = 900N_4 + 180a_1 + 150b_1 + a_0$. Finally, $2N \equiv 10(a_1 + b_1) + 2a_0 \pmod{100}$ implies that

$$60a_1 + 2a_0 \equiv 360a_1 + 300b_1 + 2a_0 \equiv 10a_1 + 10b_1 + 2a_0 \pmod{100}.$$

Therefore $5a_1 \equiv b_1 \pmod{10}$, equivalently, $b_1 \equiv 0 \pmod{5}$ and $a_1 \equiv b_1 \pmod{2}$. Conversely, if $N = 900N_4 + 180a_1 + 150b_1 + a_0$, $a_0 = b_0$, and $5a_1 \equiv b_1 \pmod{10}$, then $2N \equiv 60a_1 + 2a_0 = 10(a_1 + 5a_1) + a_0 + b_0 \equiv 10(a_1 + b_1) + (a_0 + b_0) \pmod{100}$. Because $0 \le a_1 \le 4$ and $0 \le b_1 \le 5$, it follows that there are exactly 5 different pairs (a_1, b_1), namely $(0,0)$, $(2,0)$, $(4,0)$, $(1,5)$, and $(3,5)$. Each of these can be combined with 5 different values of a_0 $(0 \le a_0 \le 4)$, to determine exactly 25 different numbers N with the required property.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M:

12a13-07

The sequence $S_1, S_2, S_3, \ldots, S_{10}$ has the property that every term beginning with the third is the sum of the previous two. That is,

$$S_n = S_{n-2} + S_{n-1} \text{ for } n \geq 3.$$

Suppose that $S_9 = 110$ and $S_7 = 42$. What is S_4?

(A) 4 **(B)** 6 **(C)** 10 **(D)** 12 **(E)** 16

2013 AMC 12A, Problem #7—

"Find the value of S_8."

Solution

Answer (C): Note that $110 = S_9 = S_7 + S_8 = 42 + S_8$, so $S_8 = 110 - 42 = 68$. Thus $68 = S_8 = S_6 + S_7 = S_6 + 42$, so $S_6 = 68 - 42 = 26$. Similarly, $S_5 = 42 - 26 = 16$, and $S_4 = 26 - 16 = 10$.

Difficulty: Medium Easy

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: F-BF. Build a function that models a relationship between two quantities.

12a13-08

Given that x and y are distinct nonzero real numbers such that $x+\frac{2}{x}=y+\frac{2}{y}$, what is xy?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 4

2013 AMC 12A, Problem #8—

"Multiply the equation by xy."

Solution

Answer (D): Multiplying the given equation by $xy \neq 0$ yields $x^2y+2y=xy^2+2x$. Thus

$$x^2y-2x-xy^2+2y=x(xy-2)-y(xy-2)=(x-y)(xy-2)=0.$$

Because $x-y\neq 0$, it follows that $xy=2$.

Difficulty: Medium

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: A-APR. Perform arithmetic operations on polynomials.

12a13-10

Let S be the set of positive integers n for which $\frac{1}{n}$ has the repeating decimal representation $0.\overline{ab} = 0.ababab\ldots$, with a and b different digits. What is the sum of the elements of S?

(A) 11 **(B)** 44 **(C)** 110 **(D)** 143 **(E)** 155

2013 AMC 12A, Problem #10—

"Find the possible values of n."

Solution

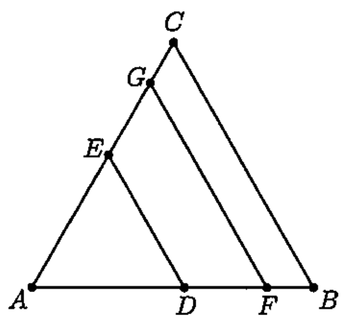
Answer (D): If n satisfies the equation $\frac{1}{n} = 0.\overline{ab}$, then $\frac{100}{n} = ab.\overline{ab}$ and subtracting gives $\frac{99}{n} = ab$. The positive factors of 99 are 1, 3, 9, 11, 33, and 99. Only $n = 11$, 33, and 99 give a number $\frac{99}{n}$ consisting of two different digits, namely 09, 03, and 01, respectively. Thus the requested sum is $11 + 33 + 99 = 143$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: N-RN. Extend the properties of exponents to rational exponents.

12a13-11

Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. What is $DE + FG$?



(A) 1 **(B)** $\frac{3}{2}$ **(C)** $\frac{21}{13}$ **(D)** $\frac{13}{8}$ **(E)** $\frac{5}{3}$

2013 AMC 12A, Problem #11—

"Use simultaneous equations to solve for DE and FG."

Solution

Answer (C): Let $x = DE$ and $y = FG$. Then the perimeter of ADE is $x+x+x = 3x$, the perimeter of $DFGE$ is $x + (y - x) + y + (y - x) = 3y - x$, and the perimeter of $FBCG$ is $y + (1 - y) + 1 + (1 - y) = 3 - y$. Because the perimeters are equal, it follows that $3x = 3y - x = 3 - y$. Solving this system yields $x = \frac{9}{13}$ and $y = \frac{12}{13}$. Thus $DE + FG = x + y = \frac{21}{13}$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT. Understand similarity in terms of similarity transformations.

12a13-12

The angles in a particular triangle are in arithmetic progression, and the side lengths are 4, 5, and x. The sum of the possible values of x equals $a+\sqrt{b}+\sqrt{c}$, where a, b, and c are positive integers. What is $a+b+c$?

(A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** 44

2013 AMC 12A, Problem #12—

"Use the Law of Cosines to find x."

Solution

Answer (A): Let the angles of the triangle be $\alpha-\delta$, α, and $\alpha+\delta$. Then $3\alpha = \alpha-\delta+\alpha+\alpha+\delta = 180°$, so $\alpha = 60°$. There are three cases depending on which side is opposite to the 60° angle. Suppose that the triangle is ABC with $\angle BAC = 60°$. Let D be the foot of the altitude from C. The triangle CAD is a 30-60-90° triangle, so $AD = \frac{1}{2}AC$ and $CD = \frac{\sqrt{3}}{2}AC$. There are three cases to consider. In each case the Pythagorean Theorem can be used to solve for the unknown side.
If $AB = 5$, $AC = 4$, and $BC = x$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = 3$. It follows that $x^2 = BC^2 = CD^2 + BD^2 = 21$, so $x = \sqrt{21}$.
If $AB = x$, $AC = 4$, and $BC = 5$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = |x-2|$. It follows that $25 = BC^2 = CD^2 + BD^2 = 12 + (x-2)^2$, and the positive solution is $x = 2+\sqrt{13}$.
If $AB = x$, $AC = 5$, and $BC = 4$, then $AD = \frac{5}{2}$, $CD = \frac{5\sqrt{3}}{2}$, and $BD = |AB - AD| = |x - \frac{5}{2}|$. It follows that $16 = BC^2 = CD^2 + BD^2 = \frac{75}{4} + (x-\frac{5}{2})^2$, which has no solution because $\frac{75}{4} > 16$.
The sum of all possible side lengths is $2+\sqrt{13}+\sqrt{21}$. The requested sum is $2+13+21 = 36$.

OR

As in the first solution, there are three cases depending on which side is opposite to the 60° angle. In each case, the Law of Cosines can be used to solve for the unknown side. If the unknown side is opposite to the 60° angle, then

$$x^2 = 4^2 + 5^2 - 2\cdot 4\cdot 5\cdot \cos(60°) = 21,$$

so $x = \sqrt{21}$.
If the side of length 5 is opposite to the 60° angle, then

$$5^2 = x^2 + 4^2 - 2\cdot 4\cdot x\cdot \cos(60°) = x^2 - 4x + 16,$$

and the positive solution is $2+\sqrt{13}$.
If the side of length 4 is opposite to the 60° angle, then

$$4^2 = x^2 + 5^2 - 2\cdot x\cdot 5\cdot \cos(60°) = x^2 - 5x + 25,$$

which has no real solutions.
The sum of all possible side lengths is $2+\sqrt{13}+\sqrt{21}$. The requested sum is $2+13+21 = 36$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

12a13-14

The sequence

$$\log_{12} 162,\ \log_{12} x,\ \log_{12} y,\ \log_{12} z,\ \log_{12} 1250$$

is an arithmetic progression. What is x?

(A) $125\sqrt{3}$ **(B)** 270 **(C)** $162\sqrt{5}$ **(D)** 434 **(E)** $225\sqrt{6}$

2013 AMC 12A, Problem #14—

"Find the value of y."

Solution

Answer (B): Because the terms form an arithmetic sequence,

$$\log_{12} y = \frac{1}{2}(\log_{12} 162 + \log_{12} 1250) = \frac{1}{2}\log_{12}(162 \cdot 1250)$$
$$= \frac{1}{2}\log_{12}(2^2 3^4 5^4) = \log_{12}(2 \cdot 3^2 5^2).$$

Then

$$\log_{12} x = \frac{1}{2}(\log_{12} 162 + \log_{12} y) = \frac{1}{2}\left(\log_{12}(2 \cdot 3^4) + \log_{12}(2 \cdot 3^2 5^2)\right)$$
$$= \frac{1}{2}\log_{12}(2^2 3^6 5^2) = \log_{12}(2 \cdot 3^3 5) = \log_{12} 270.$$

Therefore $x = 270$.

OR

If $(B_k) = (\log_{12} A_k)$ is an arithmetic sequence with common difference d, then (A_k) is a geometric sequence with common ratio $r = 12^d$. Therefore $162, x, y, z, 1250$ is a geometric sequence. Let r be their common ratio. Then $1250 = 162r^4$ and $r = \frac{5}{3}$. Thus $x = 162r = 162 \cdot \frac{5}{3} = 270$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: F-LE. Construct and compare linear, quadratic, and exponential models and solve problems.

12a13-15

Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cottontail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

(A) 96 **(B)** 108 **(C)** 156 **(D)** 204 **(E)** 372

2013 AMC 12A, Problem #15—

"Find the number of ways if Peter and Pauline are given to the same store, and if they are given to different stores."

Solution

Answer (D): There are two cases. If Peter and Pauline are given to the same pet store, then there are 4 ways to choose that store. Each of the children must then be assigned to one of the other three stores, and this can be done in $3^3 = 27$ ways. Therefore there are $4 \cdot 27 = 108$ possible assignments in this case. If Peter and Pauline are given to different stores, then there are $4 \cdot 3 = 12$ ways to choose those stores. In this case, each of the children must be assigned to one of the other two stores, and this can be done in $2^3 = 8$ ways. Therefore there are $12 \cdot 8 = 96$ possible assignments in this case. The total number of assignments is $108 + 96 = 204$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

12a13-16

A, B, and C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

(A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

2013 AMC 12A, Problem #16—

"Use simultaneous equations."

Solution

Answer (E): Let a, b, and c be the number of rocks in piles A, B, and C, respectively. Then

$$\frac{40a + 50b}{a + b} = 43 \text{ and } 7b = 3a.$$

Because 7 and 3 are relatively prime, there is a positive integer k such that $a = 7k$ and $b = 3k$. Let μ_C equal the mean weight in pounds of the rocks in C and μ_{BC} equal the mean weight in pounds of the rocks in B and C. Then

$$\frac{40 \cdot 7k + \mu_C \cdot c}{7k + c} = 44, \text{ so } \mu_C = \frac{28k + 44c}{c},$$

and

$$\mu_{BC} = \frac{50 \cdot 3k + (28k + 44c)}{3k + c} = \frac{178k + 44c}{3k + c}.$$

Clearing the denominator and rearranging yields $(\mu_{BC} - 44)c = (178 - 3\mu_{BC})k$. Because the mean weight of the rocks in the combined piles A and C is 44 pounds, and the mean weight of the rocks in B is greater than the mean weight of the rocks in A, it follows that the mean weight of the rocks in B and C must be greater than 44 pounds. Thus $(\mu_{BC} - 44)c > 0$ and therefore $178 - 3\mu_{BC}$ must be greater than zero. This implies that $\mu_{BC} < \frac{178}{3} = 59\frac{1}{3}$. If $k = 15c$ and $\mu_C = 464$, then $\mu_{BC} = 59$. Thus the greatest possible integer value for the weight in pounds of the combined piles B and C is 59.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: A-REI. Solve systems of equations.

12a13-20

Let S be the set $\{1, 2, 3, \ldots, 19\}$. For $a, b \in S$, define $a \succ b$ to mean that either $0 < a - b \leq 9$ or $b - a > 9$. How many ordered triples (x, y, z) of elements of S have the property that $x \succ y$, $y \succ z$, and $z \succ x$?

(A) 810 **(B)** 855 **(C)** 900 **(D)** 950 **(E)** 988

2013 AMC 12A, Problem #20—

"Find the number of possibilities for the entries x, y, and z."

Solution

Answer (B): Consider the elements of S as integers modulo 19. Assume $a \succ b$. If $a > b$, then $a - b \leq 9$. If $a < b$, then $b - a > 9$; that is $b - a \geq 10$ and so $(a + 19) - b \leq 9$. Thus $a \succ b$ if and only if $0 < (a - b) \pmod{19} \leq 9$.
Suppose that (x, y, z) is a triple in $S \times S \times S$ such that $x \succ y$, $y \succ z$, and $z \succ x$. There are 19 possibilities for the first entry x. Once x is chosen, y can equal $x + i$ for any i, $1 \leq i \leq 9$. Then z is at most $x + 9 + i$ and at least $x + 10$, so once y is chosen, there are i possibilities for the third element z.
The number of required triples is equal to $19(1 + 2 + \cdots + 9) = 19 \cdot \frac{1}{2} \cdot 9 \cdot 10 = 19 \cdot 45 = 855$.
Consider the elements of S as integers modulo 37. Then $a - b$ can always be taken to be in the range $-18 \leq a - b \leq 18$. Observe that $a \succ b$ if $a - b$ is positive, and $b \succ a$ if $a - b$ is negative. The first element x can be chosen freely (37 choices). Once it is chosen, we must have $y = x + i$ for some $1 \leq i \leq 18$. Then z is at most $x + 18 + i$ and at least $x + 19$, so there will be i choices for the third element of the triple.
The number of ordered triples with the desired property is $37(1 + 2 + 3 + \cdots + 18) = 37 \cdot 18 \cdot 19/2 = 6327$. The answer is $(\mathbf{B})$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

12a13-21

Consider

$$A = \log(2013 + \log(2012 + \log(2011 + \log(\cdots + \log(3 + \log 2)\cdots)))).$$

Which of the following intervals contains A?

(A) $(\log 2016, \log 2017)$
(B) $(\log 2017, \log 2018)$
(C) $(\log 2018, \log 2019)$
(D) $(\log 2019, \log 2020)$
(E) $(\log 2020, \log 2021)$

2013 AMC 12A, Problem #21—

"Find a pattern for the different values of A."

Solution

Answer (A): Let $A_n = \log(n + \log((n-1) + \log(\cdots + \log(3 + \log 2)\cdots)))$. Note that $0 < \log 2 = A_2 < 1$. If $0 < A_{k-1} < 1$, then $k < k + A_{k-1} < k+1$. Hence $0 < \log k < \log(k + A_{k-1}) = A_k < \log(k+1) \le 1$, as long as $\log k > 0$ and $\log(k+1) \le 1$, which occurs when $2 \le k \le 9$. Thus $0 < A_n < 1$ for $2 \le n \le 9$.
Because $0 < A_9 < 1$, it follows that $10 < 10 + A_9 < 11$, and so $1 = \log(10) < \log(10 + A_9) = A_{10} < \log(11) < 2$. If $1 < A_{k-1} < 2$, then $k + 1 < k + A_{k-1} < k + 2$. Hence $1 < \log(k+1) < \log(k + A_{k-1}) = A_k < \log(k+2) \le 2$, as long as $\log(k+1) > 1$ and $\log(k+2) \le 2$, which occurs when $10 \le k \le 98$. Thus $1 < A_n < 2$ for $10 \le n \le 98$.
In a similar way, it can be proved that $2 < A_n < 3$ for $99 \le n \le 997$, and $3 < A_n < 4$ for $998 \le n \le 9996$.
For $n = 2012$, it follows that $3 < A_{2012} < 4$, so $2016 < 2013 + A_{2012} < 2017$ and $\log 2016 < A_{2013} < \log 2017$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: F-IF. Analyze functions using different representations.

12a13-22

A palindrome is a nonnegative integer number that reads the same forwards and backwards when written in base 10 with no leading zeros. A 6-digit palindrome n is chosen uniformly at random. What is the probability that $\frac{n}{11}$ is also a palindrome?

(A) $\frac{8}{25}$ **(B)** $\frac{33}{100}$ **(C)** $\frac{7}{20}$ **(D)** $\frac{9}{25}$ **(E)** $\frac{11}{30}$

2013 AMC 12A, Problem #22—

"Find the possible first and last digits of n."

Solution

Answer (E): Let n be a 6-digit palindrome, $m = \frac{n}{11}$, and suppose m is a palindrome as well. First, if m is a 4-digit number, then $n = 11m < 11 \cdot 10^5 = 10^6 + 10^5$. Thus the first and last digit of n is 1. Thus the last digit of m is 1 and then the first digit of m must be 1 as well. Then $m \le 1991 < 2000$ and $n = 11m < 11 \cdot 2000 = 22\,000$, which is a contradiction. Therefore m is a 5-digit number $abcba$. If $a + b \le 9$ and $b + c \le 9$, then there are no carries in the sum $n = 11m = abcba0 + abcba$; thus the digits of n in order are a, $a+b$, $b+c$, $b+c$, $a+b$, and a. Conversely, if $a + b \ge 10$, then the first digit of n is $a+1$ and the last digit a; and if $a+b \le 9$ but $b+c \ge 10$, then the second digit of n is $a+b+1$ if $a+b < 9$, or 0 if $a+b = 9$, and the previous to last digit is $a+b$. In any case n is not a palindrome. Therefore $n = 11m$ is a palindrome if and only if $a+b \le 9$ and $b+c \le 9$. Thus the number of pairs (m, n) is equal to

$$\sum_{b=0}^{9} \sum_{c=0}^{9-b} \sum_{a=1}^{9-b} 1 = \sum_{b=0}^{9} (10-b)(9-b).$$

Letting $j = 10 - b$ gives

$$\sum_{j=1}^{10} j(j-1) = \frac{10 \cdot 11 \cdot 21}{6} - \frac{10 \cdot 11}{2} = 330.$$

The total number of 6-digit palindromes $abccba$ is determined by 10 choices for each of b and c, and 9 choices for a, for a total of $9 \cdot 10^2 = 900$. Thus the required probability is $\frac{330}{900} = \frac{11}{30}$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: S-CP. Understand independence and conditional probability and use them to interpret data.

12a13-23

$ABCD$ is a square of side length $\sqrt{3}+1$. Point P is on $\overline{AC}$ such that $AP=\sqrt{2}$. The square region bounded by $ABCD$ is rotated 90° counterclockwise with center P, sweeping out a region whose area is $\frac{1}{c}(a\pi+b)$, where a, b, and c are positive integers and $\gcd(a,b,c)=1$. What is $a+b+c$?

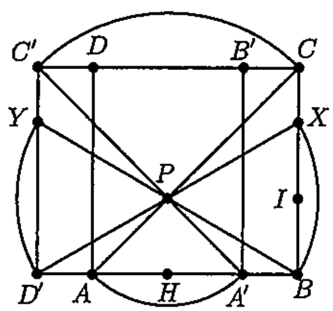
(A) 15 **(B)** 17 **(C)** 19 **(D)** 21 **(E)** 23

2013 AMC 12A, Problem #23—

"Draw a diagram of the vertices and their respective images."

Solution

Answer (C): Assume that the vertices of $ABCD$ are labeled in counterclockwise order. Let A', B', C', and D' be the images of A, B, C, and D, respectively, under the rotation. Because $\triangle A'PA$ and $\triangle C'PC$ are isosceles right triangles, points A' and C' are on lines AB and CD, respectively. Moreover, because $AP=\sqrt{2}$ and $PC=AC-AP=\sqrt{2}(\sqrt{3}+1)-\sqrt{2}=\sqrt{6}$, it follows that $AA'=\sqrt{2}AP=2$ and $CC'=\sqrt{2}CP=2\sqrt{3}$. By symmetry, points B' and D' are on lines CD and AB, respectively. Let $X\neq B$ and $Y\neq D'$ be the intersections of $\overline{BC}$ and $\overline{C'D'}$, respectively, with the circle centered at P with radius PB. Note that $PD'=PD=PB$, so this circle also contains D'. Therefore the required region consists of sectors APA', BPX, CPC', and YPD', and triangles BPA', CPX, YPC', and APD'.



Sector APA' has area $\frac{1}{4}\cdot(\sqrt{2})^2\pi=\frac{\pi}{2}$, and sector CPC' has area $\frac{1}{4}\cdot(\sqrt{6})^2\pi=\frac{3\pi}{2}$. Let H and I be the midpoints of $\overline{AA'}$ and $\overline{BX}$, respectively. Then $PH=AH=\frac{\sqrt{2}}{2}AP=1$, and $PI=HB=AB-AH=\sqrt{3}$. Thus $\triangle BPH$ is a 30-60-90° triangle, implying that $PB=2$ and $\triangle XPB$ is equilateral. Therefore congruent sectors BPX and YPD' each have area $\frac{1}{6}\cdot 2^2\pi=\frac{2\pi}{3}$.
Congruent triangles BPA' and $D'PA$ each have altitude $PH=1$ and base $A'B=AB-AH-HA'=\sqrt{3}-1$, so each has area $\frac{1}{2}(\sqrt{3}-1)$. Congruent triangles CPX and $C'PY$ each have altitude $PI=\sqrt{3}$ and base $XC=BC-BX=\sqrt{3}-1$, so each has area $\frac{1}{2}(3-\sqrt{3})$.
The area of the entire region is

$$\frac{\pi}{2}+\frac{3\pi}{2}+2\cdot\frac{2\pi}{3}+2\left(\frac{\sqrt{3}-1}{2}\right)+2\left(\frac{3-\sqrt{3}}{2}\right)=\frac{10\pi+6}{3},$$

and $a+b+c=10+6+3=19$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.

CCSS-M: G-C. Find arc lengths and areas of sectors of circles.

12a13-24

Three distinct segments are chosen at random among the segments whose endpoints are the vertices of a regular 12-gon. What is the probability that the lengths of these three segments are the three side lengths of a triangle with positive area?

(A) $\frac{553}{715}$ **(B)** $\frac{443}{572}$ **(C)** $\frac{111}{143}$ **(D)** $\frac{81}{104}$ **(E)** $\frac{223}{286}$

2013 AMC 12A, Problem #24—

"Find the subtended angle of every segment with endpoints in the 12-gon."

Solution

Answer (E): Assume without loss of generality that the regular 12-gon is inscribed in a circle of radius 1. Every segment with endpoints in the 12-gon subtends an angle of $\frac{360}{12}k = 30k$ degrees for some $1 \le k \le 6$. Let d_k be the length of those segments that subtend an angle of $30k$ degrees. There are 12 such segments of length d_k for every $1 \le k \le 5$ and 6 segments of length d_6. Because $d_k = 2\sin(15k°)$, it follows that $d_2 = 2\sin(30°) = 1$, $d_3 = 2\sin(45°) = \sqrt{2}$, $d_4 = 2\sin(60°) = \sqrt{3}$, $d_6 = 2\sin(90°) = 2$,

$$\begin{aligned} d_1 &= 2\sin(15°) = 2\sin(45° - 30°) \\ &= 2\sin(45°)\cos(30°) - 2\sin(30°)\cos(45°) = \frac{\sqrt{6}-\sqrt{2}}{2}, \text{ and} \\ d_5 &= 2\sin(75°) = 2\sin(45° + 30°) \\ &= 2\sin(45°)\cos(30°) + 2\sin(30°)\cos(45°) = \frac{\sqrt{6}+\sqrt{2}}{2}. \end{aligned}$$

If $a \le b \le c$, then $d_a \le d_b \le d_c$ and the segments with lengths d_a, d_b, and d_c do not form a triangle with positive area if and only if $d_c \ge d_a + d_b$. Because $d_2 = 1 < \sqrt{6} - \sqrt{2} = 2d_1 < \sqrt{2} = d_3$, it follows that for $(a, b, c) \in \{(1,1,3), (1,1,4), (1,1,5), (1,1,6)\}$, the segments of lengths d_a, d_b, d_c do not form a triangle with positive area. Similarly,

$$\begin{aligned} d_3 &= \sqrt{2} < \frac{\sqrt{6}-\sqrt{2}}{2} + 1 = d_1 + d_2 < \sqrt{3} = d_4, \\ d_4 < d_5 &= \frac{\sqrt{6}+\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2} + \sqrt{2} = d_1 + d_3, \text{ and} \\ d_5 < d_6 &= 2 = 1 + 1 = 2d_2, \end{aligned}$$

so for $(a, b, c) \in \{(1,2,4), (1,2,5), (1,2,6), (1,3,5), (1,3,6), (2,2,6)\}$, the segments of lengths d_a, d_b, d_c do not form a triangle with positive area. Finally, if $a \ge 2$ and $b \ge 3$, then $d_a + d_b \ge d_2 + d_3 = 1 + \sqrt{2} > 2 \ge d_c$, and also if $a \ge 3$, then $d_a + d_b \ge 2d_3 = 2\sqrt{2} > 2 = d_c$. Therefore the complete list of forbidden triples (d_a, d_b, d_c) is given by $(a, b, c) \in \{(1,1,3), (1,1,4), (1,1,5), (1,1,6), (1,2,4), (1,2,5), (1,2,6), (1,3,5), (1,3,6), (2,2,6)\}$.

For each $(a, b, c) \in \{(1,1,3), (1,1,4), (1,1,5)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 12 segments of length d_c. For each $(a, b, c) \in \{(1,1,6), (2,2,6)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 6 segments of length d_c. For each $(a, b, c) \in \{(1,2,4), (1,2,5), (1,3,5)\}$, there are 12^3 triples of segments with lengths d_a, d_b, and d_c. Finally, for each $(a, b, c) \in \{(1,2,6), (1,3,6)\}$, there are 12^2 pairs of segments with lengths d_a and d_b, and 6 segments of length d_c. Because the total number of triples of segments equals $\binom{\binom{12}{2}}{3} = \binom{66}{3}$, the required probability equals

$$\begin{aligned} &1 - \frac{3 \cdot 12 \cdot \binom{12}{2} + 2 \cdot 6 \cdot \binom{12}{2} + 3 \cdot 12^3 + 2 \cdot 12^2 \cdot 6}{\binom{66}{3}} \\ &= 1 - \frac{63}{286} = \frac{223}{286}. \end{aligned}$$

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

12a13-25

Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^2 + iz + 1$. How many complex numbers z are there such that $\text{Im}(z) > 0$ and both the real and the imaginary parts of $f(z)$ are integers with absolute value at most 10?

(A) 399 **(B)** 401 **(C)** 413 **(D)** 431 **(E)** 441

2013 AMC 12A, Problem #25—

"Find the image of the Real axis under $f(x)$."

Solution

Answer (A): Let $H = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. If $z_1, z_2 \in H$ and $f(z_1) = f(z_2)$, then $z_1^2 - z_2^2 + i(z_1 - z_2) = (z_1 - z_2)(z_1 + z_2 + i) = 0$. Because $\text{Im}(z_1) > 0$ and $\text{Im}(z_2) > 0$, it follows that $z_1 + z_2 + i \neq 0$. Thus $z_1 = z_2$; that is, the function f is one-to-one on H. Let r be a positive real number. Note that $f(r) = r^2 + 1 + ir$ describes the top part of the parabola $x = y^2 + 1$. Similarly, $f(-r) = r^2 + 1 - ir$ describes the bottom part of the parabola $x = y^2 + 1$. Because $f(i) = -1$, it follows that the image set $f(H)$ equals $\{w \in \mathbb{C} : \text{Re}(w) < (\text{Im}(w))^2 + 1\}$. Thus the set of complex numbers $w \in f(H)$ with integer real and imaginary parts of absolute value at most 10 is equal to

$$S = \{w = a + ib \in \mathbb{C} : a, b \in \mathbb{Z},\ |a| \leq 10,\ |b| \leq 10, \text{ and } a < b^2 + 1\}.$$

Because f is one-to-one, the required answer is $|f^{-1}(S)| = |S|$ and

$$|S| = 21^2 - \sum_{b=-3}^{3} \sum_{a=b^2+1}^{10} 1 = 441 - \sum_{b=-3}^{3} (10 - b^2)$$
$$= 441 - (1 + 6 + 9 + 10 + 9 + 6 + 1) = 399.$$

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M: N-CN. Perform arithmetic operations with complex numbers.

12b13-08

Line ℓ_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line ℓ_2 has equation $y = 1$ and meets line ℓ_1 at point B. Line ℓ_3 has positive slope, goes through point A, and meets ℓ_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of ℓ_3?

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** 1 **(D)** $\frac{4}{3}$ **(E)** $\frac{3}{2}$

2013 AMC 12B, Problem #8—

"Find the solution to the system of equations $3x - 2y = 1$ and $y = 1$."

Solution

Answer (B): The solution to the system of equations $3x - 2y = 1$ and $y = 1$ is $B = (x, y) = (1, 1)$. The perpendicular distance from A to $\overline{BC}$ is 3. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 3 \cdot BC = 3$, so $BC = 2$. Thus point C is 2 units to the right or to the left of $B = (1, 1)$. If $C = (-1, 1)$ then the line AC is vertical and the slope is undefined. If $C = (3, 1)$, then the line AC has slope $\frac{1-(-2)}{3-(-1)} = \frac{3}{4}$.

Difficulty: Medium

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 5. Use appropriate tools strategically; 8. Look for and express regularity in repeated reasoning.
CCSS-M: G-GPE. Use coordinates to prove simple geometric theorems algebraically.

12b13-09

What is the sum of the exponents of the prime factors of the square root of the largest perfect square that divides $12!$?

(A) 5 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 12

2013 AMC 12B, Problem #9—

"What is the prime factorization of $12!$?"

Solution

Answer (C): Because $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, the largest perfect square that divides $12!$ is $2^{10} \cdot 3^4 \cdot 5^2$ which has square root $2^5 \cdot 3^2 \cdot 5$. The sum of the exponents is $5 + 2 + 1 = 8$.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

12b13-11

Two bees start at the same spot and fly at the same rate in the following directions. Bee A travels 1 foot north, then 1 foot east, then 1 foot upwards, and then continues to repeat this pattern. Bee B travels 1 foot south, then 1 foot west, and then continues to repeat this pattern. In what directions are the bees traveling when they are exactly 10 feet away from each other?

(A) A east, B west
(B) A north, B south
(C) A north, B west
(D) A up, B south
(E) A up, B west

2013 AMC 12B, Problem #11—

"Use the x, y, and z coordinate axes correspond to the directions east, north, and up, respectively."

Solution

Answer (A): Suppose that the two bees start at the origin and that the positive directions of the x, y, and z coordinate axes correspond to the directions east, north, and up, respectively. Note that the bees are always getting farther apart from each other. After bee A has traveled 7 feet it will have gone 3 feet north, 2 feet east, and 2 feet up. Its position would be the point $(2,3,2)$. In the same time bee B will have gone 4 feet south and 3 feet west, and its position would be the point $(-3,-4,0)$. This puts them at a distance

$$\sqrt{(2-(-3))^2+(3-(-4))^2+2^2}=\sqrt{78}<10$$

After this moment, bee A will travel east to the point $(3,3,2)$ and bee B will travel west to the point $(-4,-4,0)$. Their distance after traveling one foot will be

$$\sqrt{(3-(-4))^2+(3-(-4))^2+2^2}=\sqrt{102}>10.$$

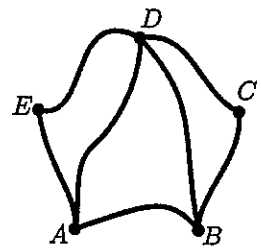
Hence bee A is traveling east and bee B is traveling west when they are exactly 10 feet away from each other.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: G-GPE. Use coordinates to prove simple geometric theorems algebraically.

12b13-12

Cities A, B, C, D, and E are connected by roads $\widetilde{AB}$, $\widetilde{AD}$, $\widetilde{AE}$, $\widetilde{BC}$, $\widetilde{BD}$, $\widetilde{CD}$, and $\widetilde{DE}$. How many different routes are there from A to B that use each road exactly once? (Such a route will necessarily visit some cities more than once.)



(A) 7 **(B)** 9 **(C)** 12 **(D)** 16 **(E)** 18

2013 AMC 12B, Problem #12—

"Simplify the diagram by replacing routes through C and E with a second A–D road and a second B–D road, respectively."

Solution

Answer (D): Cities C and E and the roads leading in and out of them can be replaced by a second A–D road and a second B–D road, respectively. If routes are designated by the list of cities they visit in order, then there are 4 types of routes: $ABDADB$, $ADABDB$, $ADBADB$, and $ADBDAB$. Each type of route represents 4 actual routes, because the trip between A and D can include the detour through E either the first or the second time, and a similar choice applies for the trip between B and D. Therefore there are $4 \cdot 4 = 16$ different routes.

Difficulty: Medium Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them.
CCSS-M:

12b13-13

The internal angles of quadrilateral $ABCD$ form an arithmetic progression. Triangles ABD and DCB are similar with $\angle DBA = \angle DCB$ and $\angle ADB = \angle CBD$. Moreover, the angles in each of these two triangles also form an arithmetic progression. In degrees, what is the largest possible sum of the two largest angles of $ABCD$?

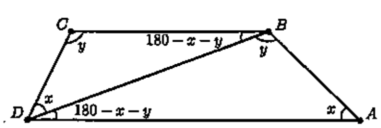
(A) 210 **(B)** 220 **(C)** 230 **(D)** 240 **(E)** 250

2013 AMC 12B, Problem #13—

"Draw a figure of quadrilateral $ABCD$."

Solution

Answer (D): Let the degree measures of the angles be as shown in the figure. The angles of a triangle form an arithmetic progression if and only if the median angle is 60°, so one of x, y, or $180 - x - y$ must be equal to 60. By symmetry of the role of the triangles ABD and DCB, assume that $x \le y$. Because $x \le y < 180 - x$ and $x < 180 - y \le 180 - x$, it follows that the arithmetic progression of the angles in $ABCD$ from smallest to largest must be either $x, y, 180 - y, 180 - x$ or $x, 180 - y, y, 180 - x$. Thus either $x + 180 - y = 2y$, in which case $3y = x + 180$; or $x + y = 2(180 - y)$, in which case $3y = 360 - x$. Neither of these is compatible with $y = 60$ (the former forces $x = 0$ and the latter forces $x = 180$), so either $x = 60$ or $x + y = 120$.



First suppose that $x = 60$. If $3y = x + 180$, then $y = 80$, and the sequence of angles in $ABCD$ is $(x, y, 180 - y, 180 - x) = (60, 80, 100, 120)$. If $3y = 360 - x$, then $y = 100$, and the sequence of angles in $ABCD$ is $(x, 180 - y, y, 180 - x) = (60, 80, 100, 120)$. Finally, suppose that $x + y = 120$. If $3y = x + 180$, then $y = 75$, and the sequence of angles in $ABCD$ is $(x, y, 180 - y, 180 - x) = (45, 75, 105, 135)$. If $3y = 360 - x$, then $y = 120$ and $x = 0$, which is impossible.
Therefore, the sum in degrees of the two largest possible angles is $105 + 135 = 240$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

12b13-16

Let $ABCDE$ be an equiangular convex pentagon of perimeter 1. The pairwise intersections of the lines that extend the sides of the pentagon determine a five-pointed star polygon. Let s be the perimeter of this star. What is the difference between the maximum and the minimum possible values of s?

(A) 0 **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{5}-1}{2}$ **(D)** $\frac{\sqrt{5}+1}{2}$ **(E)** $\sqrt{5}$

2013 AMC 12B, Problem #16—

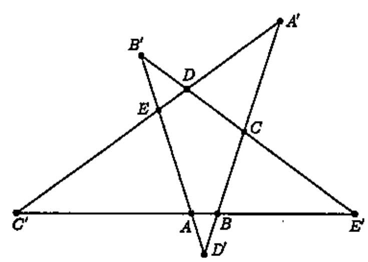
"What is the sum of the internal angles of the pentagon $ABCDE$?"

Solution

Answer (A): The sum of the internal angles of the pentagon $ABCDE$ is $3 \cdot 180° = 540°$ and by assumption all internal angles are equal, so they are all equal to $\frac{1}{5}(540°) = 108°$. Therefore the supplementary angles at each of the vertices are all equal to $180° - 108° = 72°$. It follows that all the triangles making up the points of the star are isosceles triangles with angles measuring $72°$, $72°$, and $36°$. Label the rest of the vertices of the star as in the figure. By the above argument, there is a constant c such that $A'C = A'D = c \cdot CD$ and similar expressions for the other four points of the star. Therefore the required perimeter equals

$$A'C + A'D + B'D + B'E + C'A + C'E + D'A + D'B + E'B + E'C = 2c(CD + DE + EA + AB + BC) = 2c,$$

and therefore the maximum and minimum values are the same and their difference is 0.



Note: The constant c equals $\frac{1}{2}\csc(\frac{\pi}{10}) = \frac{1}{2}(\sqrt{5}+1)$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

12b13-17

Let a, b, and c be real numbers such that

$$\begin{cases} a+b+c=2, \text{ and} \\ a^2+b^2+c^2=12. \end{cases}$$

What is the difference between the maximum and minimum possible values of c?

(A) 2 **(B)** $\frac{10}{3}$ **(C)** 4 **(D)** $\frac{16}{3}$ **(E)** $\frac{20}{3}$

2013 AMC 12B, Problem #17—

"Introduce an arbitrary real number x."

Solution

Answer (D): From the equations, $a+b=2-c$ and $a^2+b^2=12-c^2$. Let x be an arbitrary real number, then $(x-a)^2+(x-b)^2 \geq 0$; that is, $2x^2-2(a+b)x+(a^2+b^2) \geq 0$. Thus

$$2x^2-2(2-c)x+(12-c^2) \geq 0$$

for all real values x. That means the discriminant $4(2-c)^2-4\cdot 2(12-c^2) \leq 0$. Simplifying and factoring gives $(3c-10)(c+2) \leq 0$. So the range of values of c is $-2 \leq c \leq \frac{10}{3}$. Both maximum and minimum are attainable by letting $(a,b,c)=(2,2,-2)$ and $(a,b,c)=(-\frac{2}{3},-\frac{2}{3},\frac{10}{3})$. Therefore the difference between the maximum and minimum possible values of c is $\frac{10}{3}-(-2)=\frac{16}{3}$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: A-REI. Solve systems of equations.

12b13-18

Barbara and Jenna play the following game, in which they take turns. A number of coins lie on a table. When it is Barbara's turn, she must remove 2 or 4 coins, unless only one coin remains, in which case she loses her turn. When it is Jenna's turn, she must remove 1 or 3 coins. A coin flip determines who goes first. Whoever removes the last coin wins the game. Assume both players use their best strategy. Who will win when the game starts with 2013 coins and when the game starts with 2014 coins?

(A) Barbara will win with 2013 coins, and Jenna will win with 2014 coins.

(B) Jenna will win with 2013 coins, and whoever goes first will win with 2014 coins.

(C) Barbara will win with 2013 coins, and whoever goes second will win with 2014 coins.

(D) Jenna will win with 2013 coins, and Barbara will win with 2014 coins.

(E) Whoever goes first will win with 2013 coins, and whoever goes second will win with 2014 coins.

2013 AMC 12B, Problem #18—

"Consider which player can force the remaining coins to be a multiple of 5."

Solution

Answer (B): If the game starts with 2013 coins and Jenna starts, then she picks 3 coins, and then no matter how many Barbara chooses, Jenna responds by keeping the number of remaining coins congruent to 0 (mod 5). That is, she picks 3 if Barbara picks 2, and she picks 1 if Barbara picks 4. This ensures that on her last turn Jenna will leave 0 coins and thus she will win. Similarly, if Barbara starts, then Jenna can reply as before so that the number of remaining coins is congruent to 3 (mod 5). On her last turn Barbara will have 3 coins available. She is forced to remove 2 and thus Jenna will win by taking the last coin.
If the game starts with 2014 coins and Jenna starts, then she picks 1 coin and reduces the game to the previous case of 2013 coins where she wins. If Barbara starts, she selects 4 coins. Then no matter what Jenna chooses, Barbara responds by keeping the number of remaining coins congruent to 0 (mod 5). This ensures that on her last turn Barbara will leave 0 coins and win the game. Thus whoever goes first will win the game with 2014 coins.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M:

12b13-20

For $135° < x < 180°$, points $P = (\cos x, \cos^2 x)$, $Q = (\cot x, \cot^2 x)$, $R = (\sin x, \sin^2 x)$, and $S = (\tan x, \tan^2 x)$ are the vertices of a trapezoid. What is $\sin(2x)$?

(A) $2 - 2\sqrt{2}$ **(B)** $3\sqrt{3} - 6$ **(C)** $3\sqrt{2} - 5$ **(D)** $-\frac{3}{4}$ **(E)** $1 - \sqrt{3}$

2013 AMC 12B, Problem #20—

"Consider the signs of $\cos x$ and $\sin x$, respectively. Compare the absolute magnitudes of $\cos x$ and $\sin x$. Use this to estimate the other trigonometric functions."

Solution

Answer (A): Because $135° < x < 180°$, it follows that $\cos x < 0 < \sin x$ and $|\sin x| < |\cos x|$. Thus $\tan x < 0$, $\cot x < 0$, and

$$|\tan x| = \frac{|\sin x|}{|\cos x|} < 1 < \frac{|\cos x|}{|\sin x|} = |\cot x|.$$

Therefore $\cot x < \tan x$. Moreover, $\cot x = \frac{\cos x}{\sin x} < \cos x$. Thus the four vertices P, Q, R, and S are located on the parabola $y = x^2$ and P and S are in between Q and R. If $\overline{AB}$ and $\overline{CD}$ are chords on the parabola $y = x^2$ such that the x-coordinates of A and B are less than the x-coordinates of C and D, then the slope of $\overline{AB}$ is less than the slope of $\overline{CD}$. It follows that the two parallel sides of the trapezoid must be $\overline{QR}$ and $\overline{PS}$. Thus the slope of $\overline{QR}$ is equal to the slope of $\overline{PS}$. Thus,

$$\cot x + \sin x = \tan x + \cos x.$$

Multiplying by $\sin x \cos x \neq 0$ and rearranging gives the equivalent identity

$$(\cos x - \sin x)(\cos x + \sin x - \sin x \cos x) = 0.$$

Because $\cos x - \sin x \neq 0$ in the required range, it follows that $\cos x + \sin x - \sin x \cos x = 0$. Squaring and using the fact that $2 \sin x \cos x = \sin(2x)$ gives $1 + \sin(2x) = \frac{1}{4}\sin^2(2x)$. Solving this quadratic equation in the variable $\sin(2x)$ gives $\sin(2x) = 2 \pm 2\sqrt{2}$. Because $-1 < \sin 2x < 1$, the only solution is $\sin(2x) = 2 - 2\sqrt{2}$. There is indeed such a trapezoid for $x = 180° + \frac{1}{2}\arcsin(2 - 2\sqrt{2}) \approx 152.031°$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 7. Look for and make use of structure; 8. Look for and express regularity in repeated reasoning.
CCSS-M: F-TF. Model periodic phenomena with trigonometric functions.

12b13-21

Consider the set of 30 parabolas defined as follows: all parabolas have as focus the point $(0,0)$ and the directrix lines have the form $y = ax + b$ with a and b integers such that $a \in \{-2,-1,0,1,2\}$ and $b \in \{-3,-2,-1,1,2,3\}$. No three of these parabolas have a common point. How many points in the plane are on two of these parabolas?

(A) 720 **(B)** 760 **(C)** 810 **(D)** 840 **(E)** 870

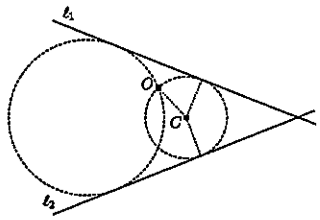
2013 AMC 12B, Problem #21—

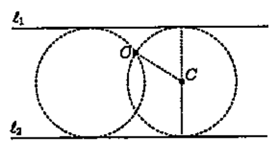
"Draw a figure for when the directrices intersect and when the directrices are parallel."

12b13-21 continued

Solution

Answer (C): If the directrices of two parabolas with the same focus intersect, then the corresponding parabolas intersect in exactly two points. The same conclusion holds if the directrices are parallel and the focus is between the two lines. Moreover, if the directrices are parallel and the focus is not between the two lines, then the corresponding parabolas do not intersect. Indeed, a point C belongs to the intersection of the parabolas with focus O and directrices ℓ_1 and ℓ_2, if and only if, $d(C, \ell_1) = OC = d(C, \ell_2)$. That is, the circle with center C and radius OC is tangent to both ℓ_1 and ℓ_2. If ℓ_1 and ℓ_2 are parallel and O is not between them, then clearly such circle does not exist. If ℓ_1 and ℓ_2 intersect and O is not on them, then there are are exactly two circles tangent to both ℓ_1 and ℓ_2 that go through O. The same is true if ℓ_1 and ℓ_2 are parallel and O is between them.

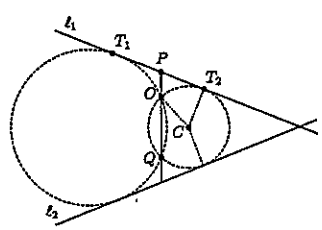




Thus there are $\binom{30}{2}$ pairs of parabolas and the pairs that do not intersect are exactly those whose directrices have the same slope and whose y-intercepts have the same sign. There are 5 different slopes and $2 \cdot \binom{3}{2} = 6$ pairs of y-intercepts with the same sign taken from $\{-3, -2, -1, 1, 2, 3\}$. Because the pairs of parabolas that intersect do so at exactly two points and no point is in three parabolas, it follows that the total number of intersection points is

$$2\left(\binom{30}{2} - 5 \cdot 6\right) = 810.$$

Note: It is possible to construct the two circles through O and tangent to the lines ℓ_1 and ℓ_2 as follows: Let ℓ' be the bisector of the angle determined by the angular sector spanned by ℓ_1 and ℓ_2 that contains O (or the midline of ℓ_1 and ℓ_2 if these lines are parallel and O is between them). Let Q be the symmetric point of O with respect to ℓ' and let P be the intersection of ℓ_1 and the line OQ (if $O = Q$ then let P be the intersection of ℓ_1 and a perpendicular line to ℓ' by O). If C is one of the desired circles, then C passes through O and Q and is tangent to ℓ_1. Let T be the point of tangency of C and ℓ_1. By the Power of a Point Theorem, $PT^2 = PO \cdot PQ$. The circle with center P and radius $\sqrt{PO \cdot PQ}$ intersects ℓ_1 in two points T_1 and T_2. The circumcircles of OQT_1 and OQT_2 are the desired circles.



Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: F-LE. Interpret expressions for functions in terms of the situation they model.

12b13-22

Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$?

(A) 12 **(B)** 20 **(C)** 24 **(D)** 48 **(E)** 272

2013 AMC 12B, Problem #22—

"Use the change of base identity to find the equivalent equation."

Solution

Answer (A): Using the change of base identity gives $\log n \cdot \log_n x = \log x$ and $\log m \cdot \log_m x = \log x$. The equivalent equation is

$$(\log x)^2 - \frac{1}{8}(7\log m + 6\log n)\log x - \frac{2013}{8}\log m \cdot \log n = 0.$$

As a quadratic equation in $\log x$, the sum of the two solutions $\log x_1$ and $\log x_2$ is equal to the negative of the linear coefficient. It follows that

$$\log(x_1 x_2) = \log x_1 + \log x_2 = \frac{1}{8}(7\log m + 6\log n) = \log\left(\left(m^7 n^6\right)^{1/8}\right).$$

Let $N = x_1 x_2$ be the product of the solutions. Suppose p is a prime dividing m. Let p^a and p^b be the largest powers of p that divide m and n respectively. Then p^{7a+6b} is the largest power of p that divides $m^7 n^6 = N^8$. It follows that $7a+6b \equiv 0 \pmod 8$. If a is odd, then there is no solution to $7a+6b \equiv 0 \pmod 8$ because $7a$ is not divisible by $\gcd(6,8) = 2$. If $a \equiv 0 \pmod 8$, then because $a > 0$, it follows that $N^8 = m^7 n^6 \geq (p^8)^7 = p^{56} \geq 2^{56}$, so $N \geq 2^7 = 128$. If $a \equiv 2 \pmod 8$ then $14 + 6b \equiv 0 \pmod 8$ is equivalent to $3b + 3 \equiv 3b + 7 \equiv 0 \pmod 4$. Thus $b \equiv 3 \pmod 4$ and then $N^8 = m^7 n^6 \geq (p^2)^7 (p^3)^6 = p^{32} \geq 2^{32}$, so $N \geq 2^4 = 16$ with equality for $m = 2^2$ and $n = 2^3$. Finally, if $a \geq 4$ and a is not a multiple of 8, then $b \geq 1$ and thus $N^8 = m^7 n^6 \geq (p^4)^7 (p^1)^6 = p^{34} \geq 2^{34}$, so $N \geq 2^{17/4} > 2^4 = 16$. Therefore the minimum product is $N = 16$ obtained uniquely when $m = 2^2$ and $n = 2^3$. The requested sum is $m + n = 4 + 8 = 12$.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: A-REI. Solve systems of equations; F-LE. Construct and compare linear, quadratic, and exponential models and solve problems.

12b13-24

Let ABC be a triangle where M is the midpoint of $\overline{AC}$, and $\overline{CN}$ is the angle bisector of $\angle ACB$ with N on $\overline{AB}$. Let X be the intersection of the median $\overline{BM}$ and the bisector $\overline{CN}$. In addition $\triangle BXN$ is equilateral and $AC = 2$. What is BN^2?

(A) $\frac{10-6\sqrt{2}}{7}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{5\sqrt{2}-3\sqrt{3}}{8}$ **(D)** $\frac{\sqrt{2}}{6}$ **(E)** $\frac{3\sqrt{3}-4}{5}$

2013 AMC 12B, Problem #24—

"Draw a very detailed and accurate figure for $\triangle ABC$."

12b13-24 continued

Solution

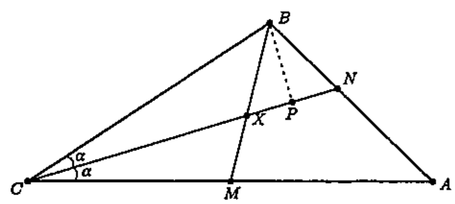
Answer (A): Let $\alpha = \angle ACN = \angle NCB$ and $x = BN$. Because $\triangle BXN$ is equilateral it follows that $\angle BXC = \angle CNA = 120°$, $\angle CBX = \angle BAC = 60° - \alpha$, and $\angle CBA = \angle BMC = 120° - \alpha$. Thus $\triangle ABC \sim \triangle BMC$ and $\triangle ANC \sim \triangle BXC$. Then

$$\frac{BC}{2} = \frac{BC}{AC} = \frac{MC}{BC} = \frac{1}{BC},$$

so $BC = \sqrt{2}$; and

$$\frac{CX + x}{2} = \frac{CN}{AC} = \frac{CX}{BC} = \frac{CX}{\sqrt{2}},$$

so $CX = (\sqrt{2} + 1)x$.



Let P be the midpoint of $\overline{XN}$. Because $\triangle BXN$ is equilateral, the triangle BPC is a right triangle with $\angle BPC = 90°$. Then by the Pythagorean Theorem,

$$\begin{aligned} 2 = BC^2 = CP^2 + PB^2 &= (CX + XP)^2 + PB^2 \\ &= \left(CX + \frac{1}{2}BN\right)^2 + \left(\frac{\sqrt{3}}{2}BN\right)^2 \\ &= \left(\sqrt{2} + \frac{3}{2}\right)^2 x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 x^2 = (5 + 3\sqrt{2})x^2. \end{aligned}$$

Therefore

$$x^2 = \frac{2}{5 + 3\sqrt{2}} = \frac{10 - 6\sqrt{2}}{7}.$$

OR

Establish as in the first solution that $CX = (\sqrt{2} + 1)x$. Then the Law of Cosines applied to $\triangle BCX$ gives

$$\begin{aligned} 2 = BC^2 &= BX^2 + CX^2 - 2BX \cdot CX \cdot \cos(120°) \\ &= x^2 + (1 + \sqrt{2})^2 x^2 + (1 + \sqrt{2})x^2 \\ &= (5 + 3\sqrt{2})x^2, \end{aligned}$$

and solving for x^2 gives the requested answer.

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 6. Attend to precision; 8. Look for and express regularity in repeated reasoning.
CCSS-M: G-SRT. Apply trigonometry to general triangles.

12b13-25

Let G be the set of polynomials of the form

$$P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_2z^2 + c_1z + 50,$$

where $c_1, c_2, \ldots, c_{n-1}$ are integers and $P(z)$ has n distinct roots of the form $a + ib$ with a and b integers. How many polynomials are in G?

(A) 288 **(B)** 528 **(C)** 576 **(D)** 992 **(E)** 1056

2013 AMC 12B, Problem #25—

"Factor the polynomial into linear and quadratic factors and observe that the constant terms must be factors of 50."

Solution

Answer (B): Let $P(z)$ be a polynomial in G. Because the coefficients of $P(z)$ are real, it follows that the nonreal roots of $P(z)$ must be paired by conjugates; that is, if $a + ib$ is a root, then $a - ib$ is a root as well. In particular, $P(z)$ can be factored into the product of pairwise different linear polynomials of the form $(z - c)$ with $c \in \mathbb{Z}$ and quadratic polynomials of the form $(z - (a + ib))(z - (a - ib)) = z^2 - 2az + (a^2 + b^2)$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Moreover, the product of the independent terms of these polynomials must be equal to 50, so each of $a^2 + b^2$ or c must be a factor of 50. Call these linear or quadratic polynomials *basic* and for every $d \in \{1, 2, 5, 10, 25, 50\}$, let B_d be the set of basic polynomials with independent term equal to $\pm d$.
The equation $a^2 + b^2 = 1$ has a pair of conjugate solutions in integers with $b \neq 0$, namely $(a, b) = (0, \pm 1)$. Thus there is only 1 basic quadratic polynomial with independent term of magnitude 1: $(z - i)(z + i) = z^2 + 1$. Similarly, the equation $a^2 + b^2 = 2$ has 2 pairs of conjugate solutions with $b \neq 0$, $(a, b) = (1, \pm 1)$ and $(-1, \pm 1)$. These give the following 2 basic polynomials with independent term ± 2: $(z - 1 - i)(z - 1 + i) = z^2 - 2z + 2$ and $(z + 1 + i)(z + 1 - i) = z^2 + 2z - 2$. In the same way the equations $a^2 + b^2 = 5$, $a^2 + b^2 = 10$, $a^2 + b^2 = 25$, and $a^2 + b^2 = 50$ have 4, 4, 5, and 6 respective pairs of conjugate solutions (a, b). These are $(2, \pm 1)$, $(-2, \pm 1)$, $(1, \pm 2)$, and $(-1, \pm 2)$; $(3, \pm 1)$, $(-3, \pm 1)$, $(1, \pm 3)$, and $(-1, \pm 3)$; $(3, \pm 4)$, $(-3, \pm 4)$, $(4, \pm 3)$, $(-4, \pm 3)$, and $(0, \pm 5)$; and $(7, \pm 1)$, $(-7, \pm 1)$, $(1, \pm 7)$, $(-1, \pm 7)$, $(5, \pm 5)$, and $(-5, \pm 5)$. These generate all possible basic quadratic polynomials with nonreal roots and independent term that divides 50. The basic linear polynomials with real roots are $z - c$ where $c \in \{\pm 1, \pm 2, \pm 5, \pm 10, \pm 25, \pm 50\}$. Thus the linear basic polynomials contribute 2 to $|B_d|$. It follows that $|B_1| = 3$, $|B_2| = 4$, $|B_5| = 6$, $|B_{10}| = 6$, $|B_{25}| = 7$, and $|B_{50}| = 8$.
Because P has independent term 50, there are either 8 choices for a polynomial in B_{50}, or $7 \cdot 4$ choices for a product of two polynomials, one in B_{25} and the other in B_2, or $6 \cdot 6$ choices for a product of two polynomials, one in B_{10} and the other in B_5, or $4 \cdot \binom{6}{2}$ choices for a product of three polynomials, one in B_2 and the other two in B_5. Finally, each of the polynomials $z + 1$ and $z^2 + 1$ in B_1 can appear or not in the product, but the presence of the polynomial $z - 1$ is determined by the rest: if the product of the remaining independent terms is -50, then it has to be present, and if the product is 50, then it must not be in the product. Thus, the grand total is

$$2^2 \left(8 + 7 \cdot 4 + 6 \cdot 6 + 4 \cdot \binom{6}{2}\right) = 2^2(8 + 28 + 36 + 60) = 4 \cdot 132 = 528.$$

Difficulty: Hard

SMP-CCSS: 1. Make sense of problems and persevere in solving them; 8. Look for and express regularity in repeated reasoning.
CCSS-M: N-RN. Extend the properties of exponents to rational exponents.